Optimal real-time transmission of Markov sources under constraints on the number of transmissions

Aditya Mahajan
Joint work with Jhelum Chakravorty

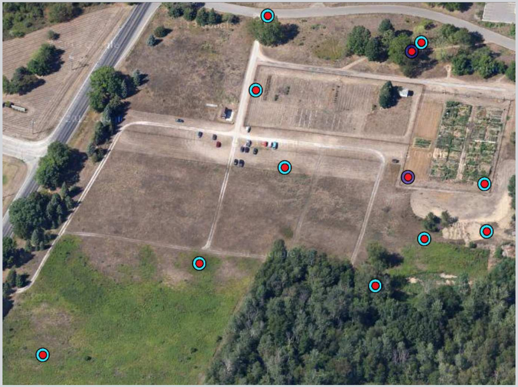
McGill University

Information Theory Seminar, University of Toronto
14 Nov, 2014

Motivation

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction

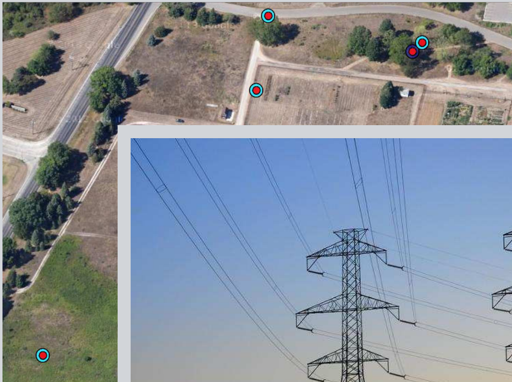
Motivation



Sensor Networks

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction

Motivation

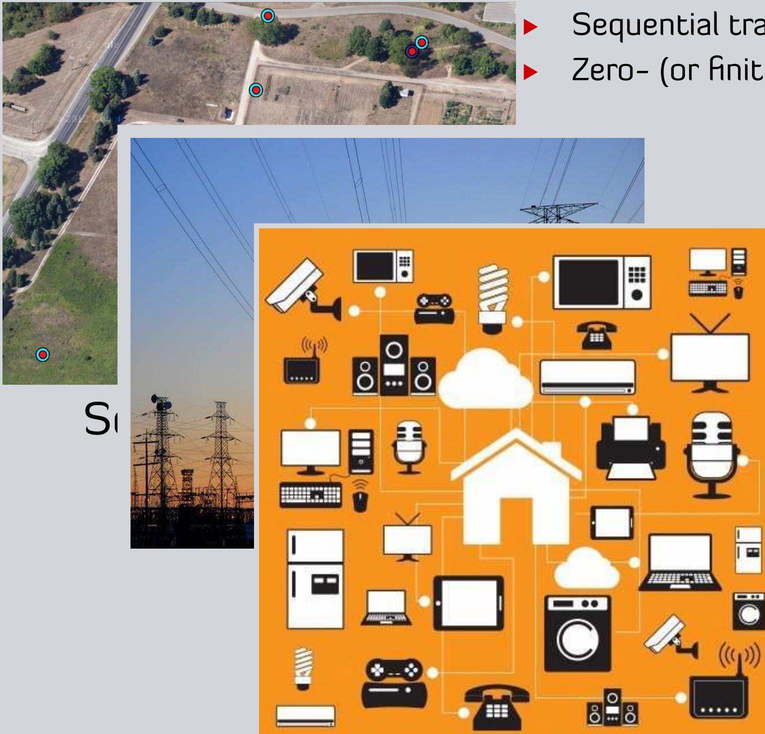


Sc

Smart Grids

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction

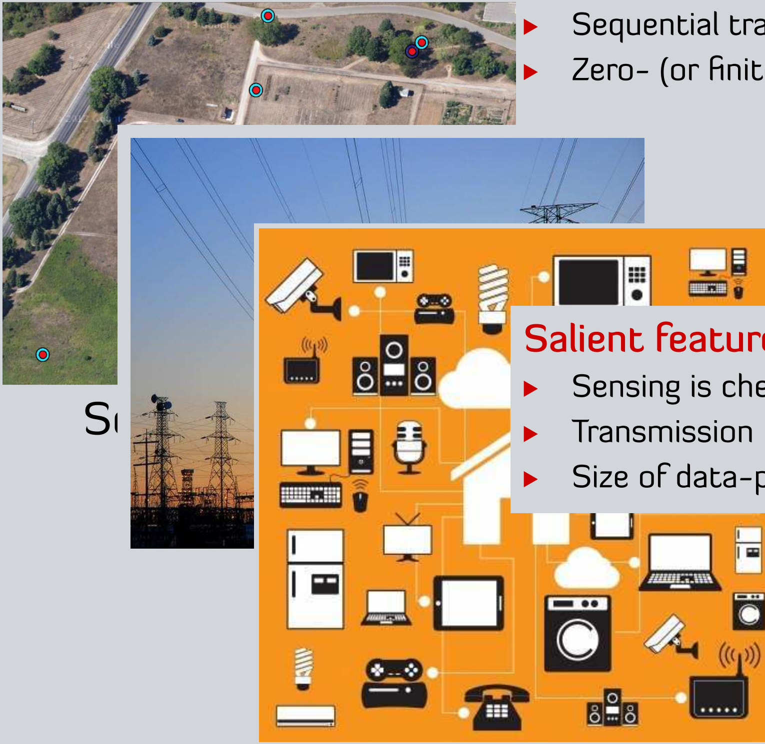
Motivation



Internet of Things

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction

Motivation



S

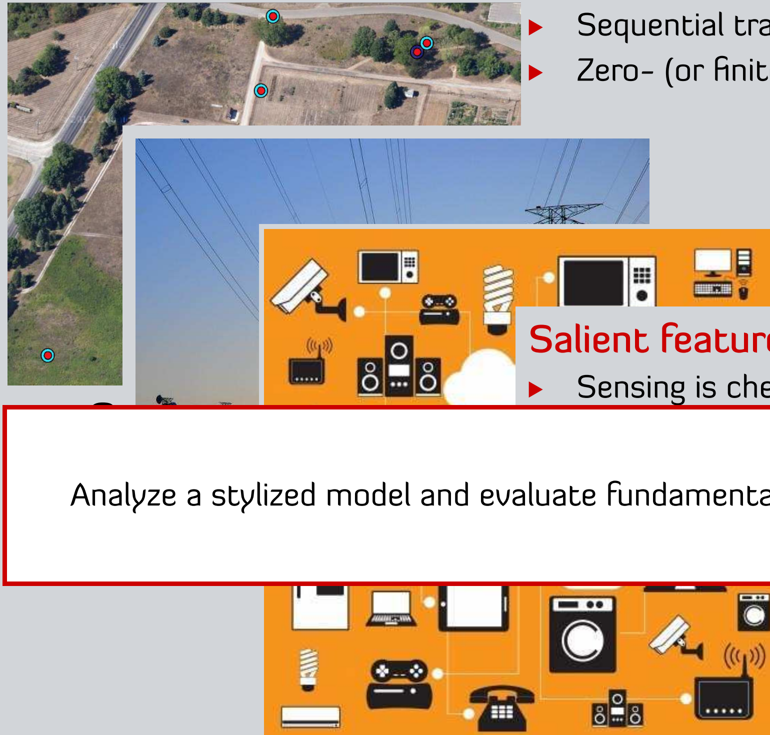
Internet of Things

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction

Salient features

- Sensing is cheap
- Transmission is expensive
- Size of data-packet is not critical

Motivation



- Sequential transmission of data
- Zero- (or finite-) delay reconstruction

Salient features

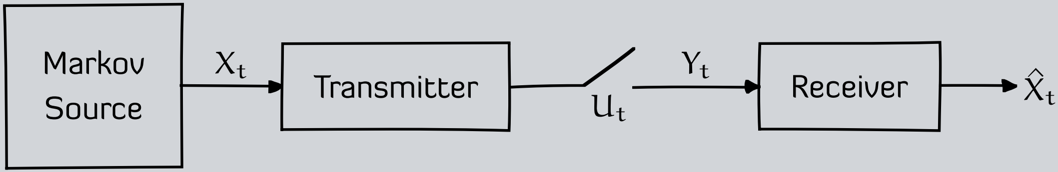
- Sensing is cheap

tical

Analyze a stylized model and evaluate fundamental tradeoffs

Internet of Things

The communication system



Source

- $X_t \in \mathbb{Z}$
- First-order time-homogeneous symmetric Markov source.

Transmitter $U_t = f_t(X_{1:t}, U_{1:t-1})$ and $Y_t = \begin{cases} X_t & \text{if } U_t = 1 \\ \varepsilon & \text{if } U_t = 0 \end{cases}$

Receiver

- $\hat{X}_t = g_t(Y_{1:t})$
- Distortion: $d(X_t - \hat{X}_t)$ where $d(e) = d(-e) \leqslant d(e+1)$

Communication Strategies

- Transmission strategy $f = \{f_t\}_{t=0}^{\infty}$.
- Estimation strategy $g = \{g_t\}_{t=0}^{\infty}$.

The constrained optimization problem

$$\min_{(f,g)} D_\beta(f,g) \quad \text{such that } N_\beta(f,g) \leqslant \alpha$$

Minimize expected distortion such that expected # of transmissions is less than α

Discounted setup

$$D_\beta(f,g) = (1-\beta)\,\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$$

$$N_\beta(f,g) = (1-\beta)\,\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t \,\Big|\, X_0 = 0\Big]$$

Average cost setup

$$D_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$$

$$N_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\Big[\sum_{t=0}^{T-1} U_t \,\Big|\, X_0 = 0\Big]$$

Assumptions on the model

(A0) $X_t \in \mathbb{Z}$, and $X_0 = 0$.

(A1) The transition matrix is Toeplitz with decaying off-diagonal terms.

$$P = \begin{bmatrix} \ddots & p_0 & \ddots & & & \\ \cdots & p_1 & p_0 & p_1 & \cdots & \\ & \ddots & p_1 & p_0 & p_1 & \cdots \\ & & \ddots & \ddots & p_0 & \ddots \end{bmatrix} \quad \text{and} \quad \begin{array}{c} p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots \\ p_0 > 0 \end{array}$$

- Nayyar et al, assumed that the transistion matrix was banded, that is, $\exists b$ such that $p_k = 0$, for all $k \geqslant b$.

(A2) The distortion function is even and increasing on $\mathbb{Z}_{\geqslant 0}$.

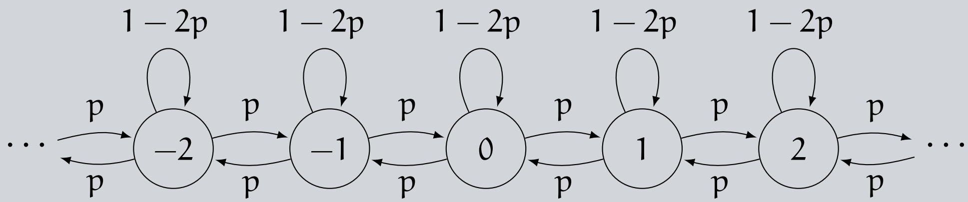
$$\forall e \in \mathbb{Z}_{\geqslant 0} : \quad d(e) = d(-e) \quad \text{and} \quad d(e) \leqslant d(e+1).$$

Furthermore,

$$d(0) = 0 \quad \text{and} \quad d(e) \neq 0, \quad \forall e \neq 0.$$

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$



Main results: Distortion-transmission function

Distortion-transmission function

$$D^*_\beta(\alpha) = \min\{D_\beta(f, g) \text{ such that } N_\beta(f, g) < \alpha\}$$

Properties:

- $D^*_\beta(\alpha)$ is convex and decreasing.
- $\lim_{\alpha\to 0} D^*_\beta(\alpha) = \infty$ and $\lim_{\alpha\to 1} D^*_\beta(\alpha) = 0$

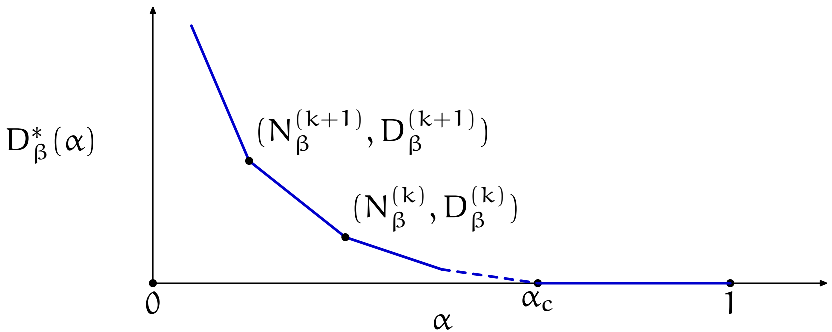
Main results: Distortion-transmission function

Distortion-transmission function

$$D^*_\beta(\alpha) = \min\{D_\beta(f, g) \text{ such that } N_\beta(f, g) < \alpha\}$$

Properties:

- $D^*_\beta(\alpha)$ is convex and decreasing.
- $\lim_{\alpha\to 0} D^*_\beta(\alpha) = \infty$ and $\lim_{\alpha\to 1} D^*_\beta(\alpha) = 0$



Main results: Optimal communication strategies

Estimation strategy

Let Z_t be the most recently transmitted symbol up to time t. Then, the optimal transmission strategy is

$$g^*(Y_{1:t}) = Z_t.$$

Transmission strategy

Let $E_t = X_t - Z_t$ and $f^{(k)}$ be a threshold-based strategy given by

$$f^{(k)}(X_t, Y_{1:t-1}) = \begin{cases} 0, & \text{if } |E_t| \leqslant k, \\ 1, & \text{if } |E_t| > k. \end{cases}$$

The optimal transmission strategy is a possibly randomized strategy that, at each stage picks

- $f^{(k^*)}$ with probability probability θ^*
- $f^{(k^*+1)}$ with probability probability $1 - \theta^*$

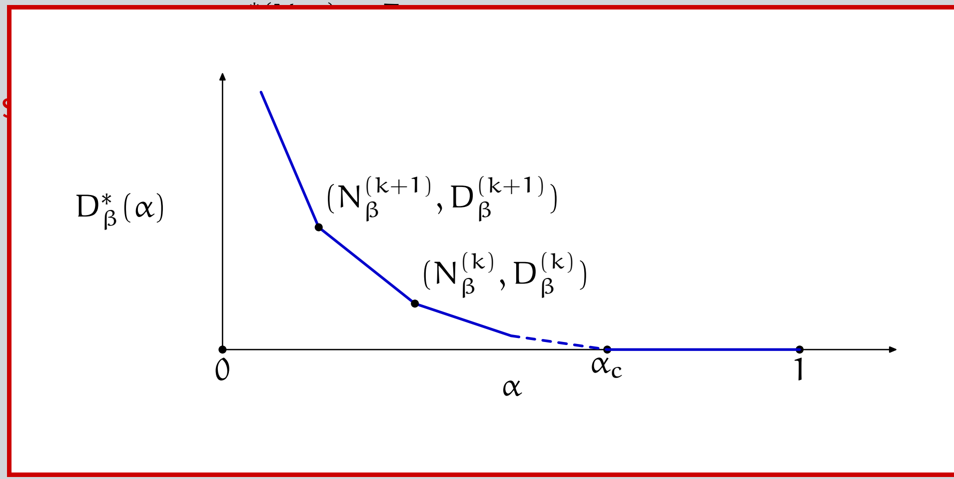
Let $N_\beta^{(k)} = N_\beta(f^{(k)}, g^*)$ and $D_\beta^{(k)} = D_\beta(f^{(k)}, g^*)$. Then:

- k^* is the largest k such that $N_\beta^{(k)} \geqslant \alpha)$
- θ^* is such that

$$\theta^* N_\beta^{(k^*)} + (1 - \theta^*) N_\beta^{(k^*+1)} = \alpha$$

Main results: Optimal communication strategies

Estimation strategy — Let Z_t be the most recently transmitted symbol up to time t. Then, the optimal transmission strategy is



Trans... given by

...nized strategy

Let $N_\beta^{(k)} = N_\beta(f^{(k)}, g^*)$ and $D_\beta^{(k)} = D_\beta(f^{(k)}, g^*)$. Then:

- k^* is the largest k such that $N_\beta^{(k)} \geqslant \alpha)$
- θ^* is such that

$$\theta^* N_\beta^{(k^*)} + (1 - \theta^*) N_\beta^{(k^*+1)} = \alpha$$

Solution methodology

Standard technique

- **Achievability**: Identify a good strategy and evaluate its performance.
- **Converse**: Determine a lower bound on distortion.

Solution methodology

Standard technique

- **Achievability**: Identify a good strategy and evaluate its performance.
- **Converse**: Determine a lower bound on distortion.
- **Hope**: The two curves match

Solution methodology

Standard technique

- **Achievability**: Identify a good strategy and evaluate its performance.
- **Converse**: Determine a lower bound on distortion.
- **Hope**: The two curves match

> Converse bounds are hard! Especially for sequential models.

Solution methodology

Standard technique

- **Achievability**: Identify a good strategy and evaluate its performance.
- **Converse**: Determine a lower bound on distortion.
- **Hope**: The two curves match

Converse bounds are hard! Especially for sequential models.

Our approach

- Model the optimization problem as a decentralized stochastic control problem. [Witsenhausen 1979, Walrand-Varaiya 1983, Teneketzis 2006, Mahajan-Teneketzis 2009, ...]
- The system has two decision makers: the transmitter and the estimator, that have access to different information.

Solution methodology

Standard technique

- **Achievability**: Identify a good strategy and evaluate its performance.
- **Converse**: Determine a lower bound on distortion.
- **Hope**: The two curves match

> Converse bounds are hard! Especially for sequential models.

Our approach

- Model the optimization problem as a decentralized stochastic control problem. [Witsenhausen 1979, Walrand-Varaiya 1983, Teneketzis 2006, Mahajan-Teneketzis 2009, . . .]
- The system has two decision makers: the transmitter and the estimator, that have access to different information.
- Identify qualitative properties of optimal strategies
- Identify a dynamic programming decomposition
- Determine optimal strategies based on the dynamic program.

Outline of the proof: Setting up a countable state DP

Step 1: Identify an information state and dynamic program for Lagrange relaxation

Use the common-information approach of Nayyar-Mahajan-Teneketzis 2013 to transform the decentralized control problem to a centralized coordination problem

Step 2: Determine qualitative properties of optimal strategies from the DP

Use majorization theory to show that, under an optimal policy, the reachable set of the information state is an almost symmetric and unimodal distribution.

The optimal transmission strategy is of a threshold-type and the optimal estimation strategy does not depend on the value of the threshold.

Previously proved by [Lipsa-Marins 2011, Nayyar-Başar-Teneketzis-Veeravalli 2013]

Step 3: Fix the estimator. Investigate the best-response transmitter

Use standard results from DP to identify sufficient conditions under which optimal strategy is time-homogeneous and given by the unique fixed-point of a DP.

Outline of the proof: Identifying a solution to the DP

Step 4: Evaluate cost of Lagrange relaxation for a particular transmission strategy

Both estimation and transmission strategies are fixed. Solve the DP to obtain renewal-theory-like relationships.

Step 5: Identify Lagrange multipliers for which a particular strategy is optimal

Similar to the idea of calibration in multi-armed bandits.

Step 6: Evaluate optimal Lagrange performance. Infer the optimal strategy for the constrained setup

The optimal Lagrange performance is continuous, piecewise linear, concave, and increasing in the Lagrange multiplier.

Show that a Bernoulli randomized simple transmission strategy is optimal.

The performance of the optimal strategy gives the distortion-transmission function.

Lagrange Relaxation

$$\min_{(f,g)} D_\beta(f, g) \quad \text{such that } N_\beta(f, g) \leqslant \alpha$$

Minimize expected distortion such that expected # of transmissions is less than α

Lagrange Relaxation

$$\min_{(f,g)} D_\beta(f, g) \quad \text{such that } N_\beta(f, g) \leqslant \alpha$$

Minimize expected distortion such that expected # of transmissions is less than α

Lagrange Relaxation

$$C^*_\beta(\lambda) \coloneqq \inf_{(f,g)} C_\beta(f, g; \lambda) \quad \text{where } C_\beta(f, g; \lambda) = D_\beta(f, g) + \lambda N_\beta(f, g)$$

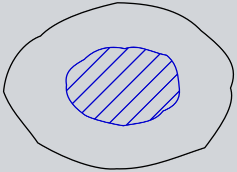
Lagrange Relaxation

$$\min_{(f,g)} D_\beta(f,g) \quad \text{such that } N_\beta(f,g) \leqslant \alpha$$

Minimize expected distortion such that expected # of transmissions is less than α

Lagrange Relaxation

$$C^*_\beta(\lambda) \coloneqq \inf_{(f,g)} C_\beta(f,g;\lambda) \quad \text{where } C_\beta(f,g;\lambda) = D_\beta(f,g) + \lambda N_\beta(f,g)$$



Search space of strategies (f,g)

- Restrict the search space of strategies (f,g) by identifying structure of optimal tranmission and estimation strategies.
- Difficulty: Non-classical information structure

Step 1a: Removing irrelevant information

Information structure $I_t = \{X_{1:t}, U_{1:t-1}, Y_{1:t-1}\}$ and $J_t = \{Y_{1:t}\}$.

Step 1a: Removing irrelevant information

Information structure $I_t = \{X_{1:t}, U_{1:t-1}, Y_{1:t-1}\}$ and $J_t = \{Y_{1:t}\}$.

Remove irrelevant data

- Aritarily fix estimation strategy.
- Finding the best-response transmitter is a centralized stochastic control problem.

- $\tilde{I}_t = \{X_t, Y_{1:t-1}\}$ is a controlled Markov process.

$$\mathbb{P}(\tilde{I}_{t+1} \mid I_t, U_t, Y_t) = \mathbb{P}(\tilde{I}_{t+1} \mid \tilde{I}_t, Y_t);$$

$$\mathbb{E}[d(X_t - \hat{X}_t) + \lambda U_t \mid I_t, U_t, Y_t] = \mathbb{E}[d(X_t - \hat{X}_t) + \lambda U_t \mid \tilde{I}_t, Y_t]$$

- Therefore, there is no loss of optimality in using control $U_t = \tilde{f}_t(\tilde{I}_t)$.

Step 1a: Removing irrelevant information

Information structure $I_t = \{X_{1:t}, U_{1:t-1}, Y_{1:t-1}\}$ and $J_t = \{Y_{1:t}\}$.

Remove irrelevant data

- Aritarily fix estimation strategy.
- Finding the best-response transmitter is a centralized stochastic control problem.
- $\tilde{I}_t = \{X_t, Y_{1:t-1}\}$ is a controlled Markov process.

$$\mathbb{P}(\tilde{I}_{t+1} \mid I_t, U_t, Y_t) = \mathbb{P}(\tilde{I}_{t+1} \mid \tilde{I}_t, Y_t);$$

$$\mathbb{E}[d(X_t - \hat{X}_t) + \lambda U_t \mid I_t, U_t, Y_t] = \mathbb{E}[d(X_t - \hat{X}_t) + \lambda U_t \mid \tilde{I}_t, Y_t]$$

- Therefore, there is no loss of optimality in using control $U_t = \tilde{f}_t(\tilde{I}_t)$.

Simplified Info Struct $\tilde{I}_t = \{X_t, Y_{1:t-1}\}$ and $J_t = \{Y_{1:t}\}$.

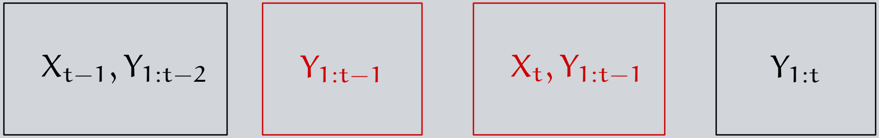
Step 1b: Equivalent centralized problem

Info Struct $\tilde{I}_t = \{X_t, Y_{1:t-1}\}$ and $J_t = \{Y_{1:t}\}$.

$X_{t-1}, Y_{1:t-2}$	$Y_{1:t-1}$	$X_t, Y_{1:t-1}$	$Y_{1:t}$

Step 1b: Equivalent centralized problem

Info Struct $\tilde{I}_t = \{X_t, Y_{1:t-1}\}$ and $J_t = \{Y_{1:t}\}$.



Coordinated system Consider a coordinator that

- observes the common-information $J_{t-1} \cap \tilde{I}_t = \{Y_{1:t-1}\}$
- chooses $(\hat{X}_{t-1}, \Gamma_t)$, where $\Gamma_t \colon X_t \mapsto U_t$.

Transmitter uses Γ_t to choose $U_t = \Gamma_t(X_t)$.

- Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Step 1b: Equivalent centralized problem

Info Struct $\tilde{I}_t = \{X_t, Y_{1:t-1}\}$ and $J_t = \{Y_{1:t}\}$.

$X_{t-1}, Y_{1:t-2}$	$Y_{1:t-1}$	$X_t, Y_{1:t-1}$	$Y_{1:t}$

Coordinated system Consider a coordinator that

- observes the common-information $J_{t-1} \cap \tilde{I}_t = \{Y_{1:t-1}\}$
- chooses $(\hat{X}_{t-1}, \Gamma_t)$, where $\Gamma_t \colon X_t \mapsto U_t$.

Transmitter uses Γ_t to choose $U_t = \Gamma_t(X_t)$.

Coordinated system is equivalent to original system

- Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Step 1c: Structural results and dynamic program

Information states

- Pre-transmission belief: $\Pi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t-1})$
- Post-transmission belief: $\Phi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t})$.

Step 1c: Structural results and dynamic program

Information states

- Pre-transmission belief: $\Pi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t-1})$
- Post-transmission belief: $\Phi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t})$.

Structural results

There is no loss of optimality in using

$$U_t = f_t(\Pi_t), \quad \text{and} \quad \hat{X}_t = g_t(\Phi_t).$$

Step 1c: Structural results and dynamic program

Information states

- Pre-transmission belief: $\Pi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t-1})$
- Post-transmission belief: $\Phi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t})$.

Structural results

There is no loss of optimality in using

$$U_t = f_t(\Pi_t), \quad \text{and} \quad \hat{X}_t = g_t(\Phi_t).$$

Dynamic program

$$W_{T+1}(\pi) = 0$$

and for $t = T, \dots, 1$

$$V_t(\varphi) = \min_{\hat{x} \in \mathcal{X}} \mathbb{E}[d(X_t, \hat{x}) + W_{t+1}(\Pi_{t+1}) \mid \Phi_t = \varphi]$$

$$W_t(\pi) = \min_{\Gamma_t} \mathbb{E}[\lambda U_t + V_t(\Phi_t) \mid \Pi_t = \pi]$$

Can we use the DP to say something more about the optimal strategy?

Qualitative properties of optimal stategies

[Imer-Başar 2005 & 2010]

Fixed number of transmissions for finite horizon LQG setup.

[Lipsa-Martins 2009 & 2011, Molin-Hirche 2009]

Remote estimation with communication cost for finite horizon LQG setup.

[Nayyar-Başar-Teneketzis-Veeravalli 2013]

Remote estimation with communication cost for finite horizon Markov chain setup. Also considered energy harvesting at the transmitter.

Step 2a: a.s.u. distributions and majorization

a.s.u. distribution A probability distribution μ over $\mathbb{Z}$ is said to be almost summetric and unimodal about a pont a if

$$\mu_{a+n} \geqslant \mu_{a-n} \geqslant \mu_{a+k+1}.$$

a.s.u. rearrangement The a.s.u. rearrangement of a probability distribution μ, denoted by μ^+ is a permutation of μ such that for every n

$$\mu^+_n \geqslant \mu^-_n \geqslant \mu^+_{n+1}$$

Majorization A probability distribution μ majorizes a distribution ν, denoted by $\mu \succeq_m \nu$ if for all n

$$\sum_{i=-k}^{k} \mu^+_i \geqslant \sum_{i=-k}^{k} \nu^+_i$$

Step 2b: Qualitative properties of optimal strategies

Monotonicity of value functions

If $\tilde{\varphi}$ is an a.s.u. distrution such that $\tilde{\varphi} \succeq_m \varphi$, then $V_t(\varphi) \geqslant V_t(\tilde{\varphi})$.

Structure of optimal estimator

If φ_t is a.s.u. about a, then the optimal estimate is a.

Structure of optimal transmitter

If π_t is a.s.u. about a, then the optimal prescription γ_t is of the form

$$\gamma_t(x) = \begin{cases} 1, & |x - a| \geqslant k(\pi_t) \\ 0, & |x - a| < k(\pi_t) \end{cases}$$

- Nayyar, Başar, Teneketzis, Veeravalli, TAC 2013

Step 2b: Qualitative properties of optimal strategies

Monotonicity of value functions

If $\tilde{\varphi}$ is an a.s.u. distrution such that $\tilde{\varphi} \succeq_m \varphi$, then $V_t(\varphi) \geqslant V_t(\tilde{\varphi})$.

Structure of optimal estimator

If φ_t is a.s.u. about a, then the optimal estimate is a.

Structure of optimal transmitter

If π_t is a.s.u. about a, then the optimal prescription γ_t is of the form

$$\gamma_t(x) = \begin{cases} 1, & |x - a| \geqslant k(\pi_t) \\ 0, & |x - a| < k(\pi_t) \end{cases}$$

Using these properties, one can show that under an optimal strategy π_t and φ_t are a.s.u.

- Nayyar, Başar, Teneketzis, Veeravalli, TAC 2013

Step 2c: Structure of optimal estimator (Nayyar et al, 2013)

Transmitted Process

Let Z_t denote the most recently transmitted value of the Markov source.

$$Z_0 = 0 \quad \text{and} \quad Z_t = \begin{cases} X_t & \text{if } U_t = 1; \\ Z_{t-1} & \text{if } U_t = 0. \end{cases}$$

The estimator can keep track of Z_t as follows:

$$Z_0 = 0 \quad \text{and} \quad Z_t = \begin{cases} Y_t & \text{if } Y_t \neq \varepsilon; \\ Z_{t-1} & \text{if } Y_t = \varepsilon. \end{cases}$$

Step 2c: Structure of optimal estimator (Nayyar et al, 2013)

Transmitted Process Let Z_t denote the most recently transmitted value of the Markov source.

$$Z_0 = 0 \quad \text{and} \quad Z_t = \begin{cases} X_t & \text{if } U_t = 1; \\ Z_{t-1} & \text{if } U_t = 0. \end{cases}$$

The estimator can keep track of Z_t as follows:

$$Z_0 = 0 \quad \text{and} \quad Z_t = \begin{cases} Y_t & \text{if } Y_t \neq \varepsilon; \\ Z_{t-1} & \text{if } Y_t = \varepsilon. \end{cases}$$

Theorem 1 The process $\{Z_t\}_{t=0}^{\infty}$ is a sufficient statistic at the estimator and an optimal estimation strategy is given by

$$\hat{X}_t = g_t^*(Z_t) = Z_t \qquad (\star)$$

Remark

- The optimal estimation strategy is time-homogeneous and can be specified in closed form.

Step 2d: Structure of optimal transmitter (Nayyar et al)

Error process Let $E_t = X_t - Z_{t-1}$ denote the error process. $\{E_t\}_{t=0}^{\infty}$ is a controlled Markov process where

$$E_0 = 0 \quad \text{and} \quad \mathbb{P}(E_{t+1} = n \mid E_t = e, U_t = u) = \begin{cases} P_{0n}, & \text{if } u = 1; \\ P_{en}, & \text{if } u = 0. \end{cases}$$

Step 2d: Structure of optimal transmitter (Nayyar et al)

Error process Let $E_t = X_t - Z_{t-1}$ denote the error process. $\{E_t\}_{t=0}^{\infty}$ is a controlled Markov process where

$$E_0 = 0 \quad \text{and} \quad \mathbb{P}(E_{t+1} = n \mid E_t = e, U_t = u) = \begin{cases} P_{0n}, & \text{if } u = 1; \\ P_{en}, & \text{if } u = 0. \end{cases}$$

Theorem 2 When the estimation strategy is of the form ($\star$), then $\{E_t\}_{t=0}^{\infty}$ is a sufficient statistic at the transmitter.

Furthermore, an optimal transmission strategy is characterized by a time-varying threshold $\{k_t\}_{t=0}^{\infty}$, i.e.,

$$U_t = f_t(E_t) = \begin{cases} 1 & \text{if } |E_t| \geq k_t; \\ 0 & \text{if } |E_t| < k_t. \end{cases}$$

Step 2d: Structure of optimal transmitter (Nayyar et al)

Error process Let $E_t = X_t - Z_{t-1}$ denote the error process. $\{E_t\}_{t=0}^{\infty}$ is a controlled Markov process where

$$E_0 = 0 \quad \text{and} \quad \mathbb{P}(E_{t+1} = n \mid E_t = e, U_t = u) = \begin{cases} P_{0n}, & \text{if } u = 1; \\ P_{en}, & \text{if } u = 0. \end{cases}$$

Theorem 2 When the estimation strategy is of the form ($\star$), then $\{E_t\}_{t=0}^{\infty}$ is a sufficient statistic at the transmitter.

Furthermore, an optimal transmission strategy is characterized by a time-varying threshold $\{k_t\}_{t=0}^{\infty}$, i.e.,

$$U_t = f_t(E_t) = \begin{cases} 1 & \text{if } |E_t| \geqslant k_t; \\ 0 & \text{if } |E_t| < k_t. \end{cases}$$

Proof idea
- The proof of [Nayyar et al, 2013] was based on some majorization inequalities of [Hajek et al, 2009] for distributions with finite support.
- We extend these inequalities to distributions over integers using results of [Wang-Woo-Madiman, 2014].

We have identified the structure of optimal transmission and estimation strategies for the finite-horizon Lagrange relaxation of the original problem.

How do these results extend to infinite horizon setup?

Step 3: Infinite horizon setup (for Lagrange relaxation)

Main idea

- Based on Thm 1, restrict attention to time-homogeneous estimation strategy

$$\hat{X}_t = g^*_t(Z_t) = Z_t$$

- Consider the problem of finding the "best response" estimation strategy.

Step 3: Infinite horizon setup (for Lagrange relaxation)

Main idea

- Based on Thm 1, restrict attention to time-homogeneous estimation strategy

$$\hat{X}_t = g^*_t(Z_t) = Z_t$$

- Consider the problem of finding the "best response" estimation strategy.

- Centralized stochastic control problem with countable state space and unbounded cost.
- Standard MDP results apply under mild technical assumptions.

Step 3: Infinite horizon setup (for Lagrange relaxation)

Main idea

- Based on Thm 1, restrict attention to time-homogeneous estimation strategy

$$\hat{X}_t = g^*_t(Z_t) = Z_t$$

- Consider the problem of finding the "best response" estimation strategy.

- Centralized stochastic control problem with countable state space and unbounded cost.
- Standard MDP results apply under mild technical assumptions.

Assum (A3) For every $\lambda \geqslant 0$, there exists a function $w : \mathbb{Z} \to \mathbb{R}$ and postive and finite constants μ_1 and μ_2 such that for all $e \in \mathbb{Z}$, we have that

$$\max\{\lambda, d(e)\} \leqslant \mu_1 w(e)$$

$$\max\left\{ \sum_{n=-\infty}^{\infty} P_{en} w(n), \sum_{n=-\infty}^{\infty} P_{0n} w(n) \right\} \leqslant \mu_2 w(e).$$

Step 3: Structure of optimal transmitter for infinite horizon

Structure Under assumption (A3), optimal transmission strategy is characterized by time-homogeneous threshold k, i.e.,

$$U_t = f(E_t) = \begin{cases} 1 & \text{if } |E_t| \geqslant k; \\ 0 & \text{if } |E_t| < k. \end{cases}$$

Step 3: Structure of optimal transmitter for infinite horizon

Structure Under assumption (A3), optimal transmission strategy is characterized by time-homogeneous threshold k, i.e.,

$$U_t = f(E_t) = \begin{cases} 1 & \text{if } |E_t| \geqslant k; \\ 0 & \text{if } |E_t| < k. \end{cases}$$

Dynamic program For $\beta \in (0,1)$, the optimal strategy is determined by the unique fixed point of the following DP:

$$V_\beta(e;\lambda) = \min\Big\{(1-\beta)\lambda + \beta \sum_{n=-\infty}^{\infty} P_{0n} V_\beta(n;\lambda), \quad \text{Transmit}$$

$$(1-\beta)d(e) + \beta \sum_{n=-\infty}^{\infty} P_{en} V_\beta(n;\lambda)\Big\} \quad \text{Don't Transmit}$$

Step 3: Structure of optimal transmitter for infinite horizon

Structure Under assumption (A3), optimal transmission strategy is characterized by time-homogeneous threshold k, i.e.,

$$U_t = f(E_t) = \begin{cases} 1 & \text{if } |E_t| \geqslant k; \\ 0 & \text{if } |E_t| < k. \end{cases}$$

Dynamic program For $\beta \in (0, 1)$, the optimal strategy is determined by the unique fixed point of the following DP:

$$V_\beta(e; \lambda) = \min \Big\{ (1-\beta)\lambda + \beta \sum_{n=-\infty}^{\infty} P_{0n} V_\beta(n; \lambda), \quad \text{Transmit}$$

$$(1-\beta) d(e) + \beta \sum_{n=-\infty}^{\infty} P_{en} V_\beta(n; \lambda) \Big\} \quad \text{Don't Transmit}$$

Lagrange relaxation Let $f^*_\beta(\cdot; \lambda)$ be the time-homogeneous optimal transmission strategy.

$$C^*_\beta(\lambda) \coloneqq \inf_{(f,g)} C_\beta(f, g; \lambda) = C_\beta(f^*_\beta, g^*; \lambda) = V_\beta(0; \lambda)$$

Step 3: The SEN Cond. and the long-term average setup

SEN Conditions

For any $\lambda \geqslant 0$, the value function $V_\beta(\cdot;\lambda)$ satisfy the SEN condition:

(S1) There exists a reference state $e_0 \in \mathbb{Z}$ such that $V_\beta(e_0;\lambda) < \infty$ for all $\beta \in (0,1)$.

(S2) Define $h_\beta(e;\lambda) = (1-\beta)^{-1}[V_\beta(e;\lambda) - V_\beta(e_0;\lambda)]$. There exists a function $K_\lambda : \mathbb{Z} \to \mathbb{R}$ such that $h_\beta(e;\lambda) \leqslant K_\lambda(e)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

(S3) There exists a non-negative (finite) constant L_λ such that $-L_\lambda \leqslant h_\beta(e;\lambda)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

Step 3: The SEN Cond. and the long-term average setup

SEN Conditions

For any $\lambda \geqslant 0$, the value function $V_\beta(\cdot;\lambda)$ satisfy the SEN condition:

(S1) There exists a reference state $e_0 \in \mathbb{Z}$ such that $V_\beta(e_0;\lambda) < \infty$ for all $\beta \in (0,1)$.

(S2) Define $h_\beta(e;\lambda) = (1-\beta)^{-1}[V_\beta(e;\lambda) - V_\beta(e_0;\lambda)]$. There exists a function $K_\lambda : \mathbb{Z} \to \mathbb{R}$ such that $h_\beta(e;\lambda) \leqslant K_\lambda(e)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

(S3) There exists a non-negative (finite) constant L_λ such that $-L_\lambda \leqslant h_\beta(e;\lambda)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

Proof ideas

- The Markov chain induced by $f^{(0)}$ is 0-standard, i.e., for every state e, the expected time and expected cost for first passage to 0 is finite.
- Hence, (S1) and (S2) hold.
- For any $e \in \mathbb{Z}_{\geqslant 0}$, $[P]_{e+1} \succ_r [P]_e$, where $\succ_r$ denotes reflected stochastic dominance.
- Using induction show that for every λ, $V_\beta(e;\lambda)$ is even and increasing in e.
- Hence (S3) holds.

Step 3: The SEN Cond. and the long-term average setup

SEN Conditions

For any $\lambda \geqslant 0$, the value function $V_\beta(\cdot;\lambda)$ satisfy the SEN condition:

(S1) There exists a reference state $e_0 \in \mathbb{Z}$ such that $V_\beta(e_0;\lambda) < \infty$ for all $\beta \in (0,1)$.

(S2) Define $h_\beta(e;\lambda) = (1-\beta)^{-1}[V_\beta(e;\lambda) - V_\beta(e_0;\lambda)]$. There exists a function $K_\lambda : \mathbb{Z} \to \mathbb{R}$ such that $h_\beta(e;\lambda) \leqslant K_\lambda(e)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

(S3) There exists a non-negative (finite) constant L_λ such that $-L_\lambda \leqslant h_\beta(e;\lambda)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

Vanishing discount approach

Let $f_1^*(\cdot;\lambda)$ be any limit point of $f_\beta^*(\cdot;\lambda)$ as $\beta \uparrow 1$.
Then the time-homogeneous transmission strategy $f_1^*(\cdot;\lambda)$ is optimal for $\beta = 1$ (the long-term average setup).

Furthermore, the performance of this optimal strategy is

$$C_1^*(\lambda) \coloneqq \inf_{(f,g)} C_1(f,g;\lambda) = C_1(f_1^*, g^*;\lambda) = \lim_{\beta\uparrow 1} V_\beta(0;\lambda) = \lim_{\beta\uparrow 1} C_\beta^*(\lambda).$$

Time-homogeneous threshold-based transmission strategies are optimal.

The optimal threshold can be determined by solving a DP.

The DP is well-behaved. The long-term average setup may be analyzed using the vanishing discount approach.

So what? Does the DP give any insights beyond numerical computations?

Step 4: Performance of a threshold based strategy

Threshold-based strategy

We analyze the performace of $(f^{(k)}, g^*)$, where

$$f^{(k)}(e) \coloneqq \begin{cases} 1, & \text{if } |e| \geqslant k; \\ 0, & \text{if } |e| < k. \end{cases}$$

Step 4: Performance of a threshold based strategy

Threshold-based strategy

We analyze the performace of $(f^{(k)}, g^*)$, where

$$f^{(k)}(e) \coloneqq \begin{cases} 1, & \text{if } |e| \geqslant k; \\ 0, & \text{if } |e| < k. \end{cases}$$

Performance of a given strategy

- Distortion $D_\beta^{(k)}(e)$ under strategy $(f^{(k)}, g^*)$:

$$D_\beta^{(k)}(e) = \begin{cases} \beta \sum_{n=-\infty}^{\infty} P_{0n} D_\beta^{(k)}(n), & |e| \geqslant k \\ (1-\beta) d(e) + \beta \sum_{n=-\infty}^{\infty} P_{en} D_\beta^{(k)}(n), & |e| < k \end{cases}$$

- Transmissions $N_\beta^{(k)}(e)$ under strategy $(f^{(k)}, g^*)$:

$$N_\beta^{(k)}(e) = \begin{cases} (1-\beta) + \beta \sum_{n=-\infty}^{\infty} P_{0n} N_\beta^{(k)}(n), & |e| \geqslant k \\ \beta \sum_{n=-\infty}^{\infty} P_{en} N_\beta^{(k)}(n), & |e| < k \end{cases}$$

Step 4: Performance of a threshold based strategy (cont.)

Cost until first transmission

Let $S^{(k)} = \{e \in \mathbb{Z} : |e| \leqslant k-1\}$ and $\tau^{(k)}$ be escape time of set $S^{(k)}$.

Step 4: Performance of a threshold based strategy (cont.)

Cost until first transmission

Let $S^{(k)} = \{e \in \mathbb{Z} : |e| \leqslant k - 1\}$ and $\tau^{(k)}$ be escape time of set $S^{(k)}$.

Define
$$L_\beta^{(k)} := \mathbb{E}\left[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \middle| E_0 = 0 \right]$$

$$M_\beta^{(k)} := \frac{1 - \mathbb{E}[\beta^{\tau^{(k)}} \mid E_0 = 0]}{1 - \beta}$$

Step 4: Performance of a threshold based strategy (cont.)

Cost until first transmission

Let $S^{(k)} = \{e \in \mathbb{Z} : |e| \leqslant k-1\}$ and $\tau^{(k)}$ be escape time of set $S^{(k)}$.

Define
$$L_\beta^{(k)} \coloneqq \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = 0\Big]$$

$$M_\beta^{(k)} \coloneqq \frac{1 - \mathbb{E}[\beta^{\tau^{(k)}} \mid E_0 = 0]}{1-\beta}$$

Renewal relationships

$$D_\beta^{(k)} = \frac{L_\beta^{(k)}}{M_\beta^{(k)}} \quad \text{and} \quad N_\beta^{(k)} = \frac{1}{M_\beta^{(k)}} - (1-\beta)$$

We show that these expressions satisfy the recursive relationships shown on the previous slide.

Step 4: Performance of a threshold based strategy (cont.)

Cost until first transmission

Let $S^{(k)} = \{e \in \mathbb{Z} : |e| \leqslant k-1\}$ and $\tau^{(k)}$ be escape time of set $S^{(k)}$.

Define $$L_\beta^{(k)} := \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = 0\Big]$$

$$M_\beta^{(k)} := \frac{1 - \mathbb{E}[\beta^{\tau^{(k)}} \mid E_0 = 0]}{1-\beta}$$

Renewal relationships

$$D_\beta^{(k)} = \frac{L_\beta^{(k)}}{M_\beta^{(k)}} \quad \text{and} \quad N_\beta^{(k)} = \frac{1}{M_\beta^{(k)}} - (1-\beta)$$

Vanishing discount relationships

$$L_1^{(k)} = \lim_{\beta\uparrow 1} L_\beta^{(k)}, \quad M_1^{(k)} = \lim_{\beta\uparrow 1} M_\beta^{(k)}.$$

and

$$D_1^{(k)} = \lim_{\beta\uparrow 1} D_\beta^{(k)} = \frac{L_1^{(k)}}{M_1^{(k)}}$$

$$N_1^{(k)} = \lim_{\beta\uparrow 1} N_\beta^{(k)} = \frac{1}{M_1^{(k)}}$$

Step 4: Computing performance

Analytic expressions for performace

Let $P^{(k)}$ and $Q_\beta^{(k)}$ be square matrices and $d^{(k)}$ is a column vector indexed by $S^{(k)}$ defined as follows:

$$P_{ij}^{(k)} \coloneqq P_{ij}, \quad \forall i, j \in S^{(k)},$$

$$Q_\beta^{(k)} \coloneqq [I_{2k-1} - \beta P^{(k)}]^{-1},$$

$$d^{(k)} \coloneqq [d(-k+1), \dots, d(k-1)]^\top$$

Then,

$$L_\beta^{(k)} = \langle [Q_\beta^{(k)}]_0, d^{(k)} \rangle \quad \text{and} \quad M_\beta^{(k)} = \langle [Q_\beta^{(k)}]_0, \mathbf{1}_{2k-1} \rangle.$$

Step 4: Computing performance

Analytic expressions for performace

Let $P^{(k)}$ and $Q_\beta^{(k)}$ be square matrices and $d^{(k)}$ is a column vector indexed by $S^{(k)}$ defined as follows:

$$P_{ij}^{(k)} \coloneqq P_{ij}, \quad \forall i, j \in S^{(k)},$$

$$Q_\beta^{(k)} \coloneqq [I_{2k-1} - \beta P^{(k)}]^{-1},$$

$$d^{(k)} \coloneqq [d(-k+1), \dots, d(k-1)]^\top$$

Then,

$$L_\beta^{(k)} = \langle [Q_\beta^{(k)}]_0, d^{(k)} \rangle \quad \text{and} \quad M_\beta^{(k)} = \langle [Q_\beta^{(k)}]_0, \mathbf{1}_{2k-1} \rangle.$$

Proof Standard Markov chain analysis.

Step 4: Computing performance

Analytic expressions for performace

Let $P^{(k)}$ and $Q_\beta^{(k)}$ be square matrices and $d^{(k)}$ is a column vector indexed by $S^{(k)}$ defined as follows:

$$P_{ij}^{(k)} \coloneqq P_{ij}, \quad \forall i, j \in S^{(k)},$$

$$Q_\beta^{(k)} \coloneqq [I_{2k-1} - \beta P^{(k)}]^{-1},$$

$$d^{(k)} \coloneqq [d(-k+1), \dots, d(k-1)]^\top$$

Then,

$$L_\beta^{(k)} = \langle [Q_\beta^{(k)}]_0, d^{(k)} \rangle \quad \text{and} \quad M_\beta^{(k)} = \langle [Q_\beta^{(k)}]_0, \mathbf{1}_{2k-1} \rangle.$$

$D_\beta^{(k)}$ and $N_\beta^{(k)}$ can be computed using these expressions.

We found performance of a generic (threshold-based) strategy.

How does this lead to identifying an optimal strategy?

Use the idea of calibration from multi-armed bandits.

Instead of finding the best strategy for a particular λ, we identify the set of λ that are optimal for a particular strategy.

Step 5: Optimal stategy for the Lagrange relaxation

Some inequalities $$L_\beta^{(k)} < L_\beta^{(k+1)}, \quad M_\beta^{(k)} < M_\beta^{(k+1)}, \quad D_\beta^{(k)} < D_\beta^{(k+1)}.$$

Step 5: Optimal stategy for the Lagrange relaxation

Some inequalities $\quad L_\beta^{(k)} < L_\beta^{(k+1)}, \quad M_\beta^{(k)} < M_\beta^{(k+1)}, \quad D_\beta^{(k)} < D_\beta^{(k+1)}.$

Proof idea

- For the first two inequalities, express $P^{(k+1)}$ in terms in $P^{(k)}$.
- For the third inequality, define operator $T^{(k+1)}$ as

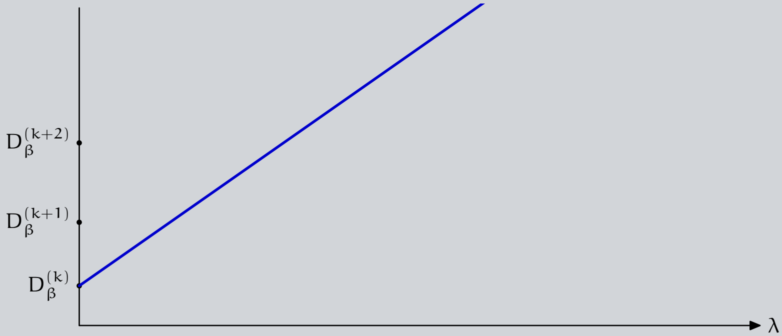
$$[T^{(k+1)}D](e) = \begin{cases} \beta \sum_{n=-\infty}^{\infty} P_{0n} D(n), & |e| \geqslant k+1 \\ (1-\beta)d(e) + \beta \sum_{n=-\infty}^{\infty} P_{en} D(n), & |e| < k+1 \end{cases}$$

  - Define $D^{(k,0)} = D^{(k)}$ and $D^{(k,m+1)} = T^{(k+1)} D^{(k,m)}$.
  - Show that $D^{(k,m)}(e) > D^{(k)}(e)$ for all $e \in A^{(m)}$, where $A^{(m)} \uparrow \mathbb{Z}$.

Step 5: Optimal stategy for the Lagrange relaxation

Some inequalities $$L_\beta^{(k)} < L_\beta^{(k+1)}, \quad M_\beta^{(k)} < M_\beta^{(k+1)}, \quad D_\beta^{(k)} < D_\beta^{(k+1)}.$$

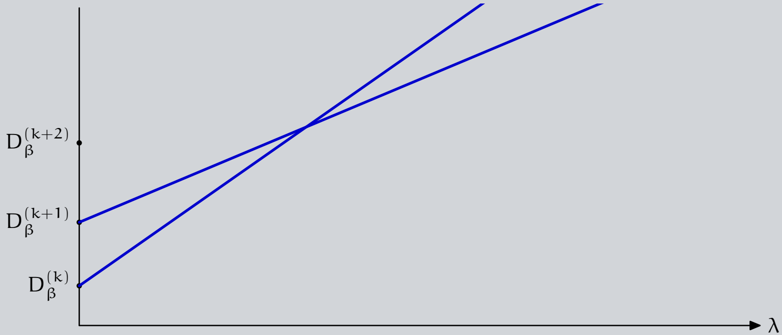
Lagrangian cost $$C_\beta^{(k)}(\lambda) \coloneqq C(f^{(k)}, g^*; \lambda) = D_\beta^{(k)} + \lambda N_\beta^{(k)}$$



Step 5: Optimal stategy for the Lagrange relaxation

Some inequalities $L_\beta^{(k)} < L_\beta^{(k+1)}, \quad M_\beta^{(k)} < M_\beta^{(k+1)}, \quad D_\beta^{(k)} < D_\beta^{(k+1)}.$

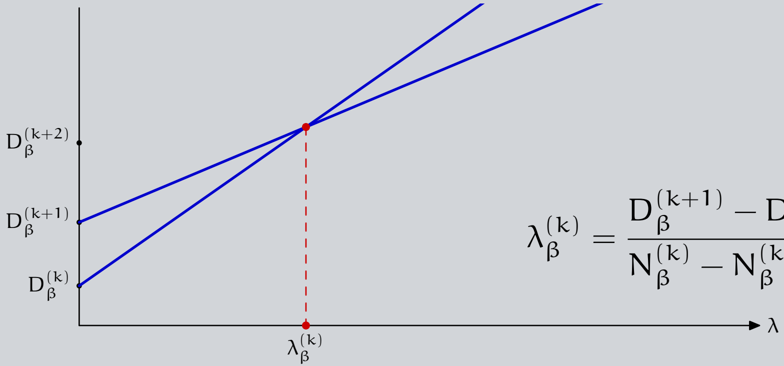
Lagrangian cost $C_\beta^{(k)}(\lambda) \coloneqq C(f^{(k)}, g^*; \lambda) = D_\beta^{(k)} + \lambda N_\beta^{(k)}$



Step 5: Optimal stategy for the Lagrange relaxation

Some inequalities $L_\beta^{(k)} < L_\beta^{(k+1)}, \quad M_\beta^{(k)} < M_\beta^{(k+1)}, \quad D_\beta^{(k)} < D_\beta^{(k+1)}.$

Lagrangian cost $C_\beta^{(k)}(\lambda) \coloneqq C(f^{(k)}, g^*; \lambda) = D_\beta^{(k)} + \lambda N_\beta^{(k)}$

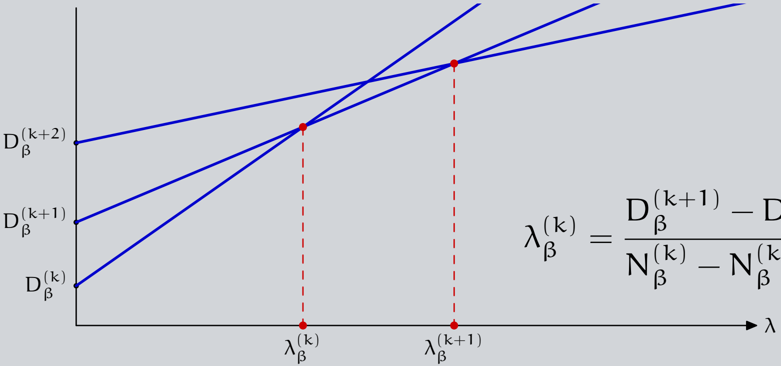


$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

Step 5: Optimal stategy for the Lagrange relaxation

Some inequalities $$L_\beta^{(k)} < L_\beta^{(k+1)}, \quad M_\beta^{(k)} < M_\beta^{(k+1)}, \quad D_\beta^{(k)} < D_\beta^{(k+1)}.$$

Lagrangian cost $$C_\beta^{(k)}(\lambda) \coloneqq C(f^{(k)}, g^*; \lambda) = D_\beta^{(k)} + \lambda N_\beta^{(k)}$$

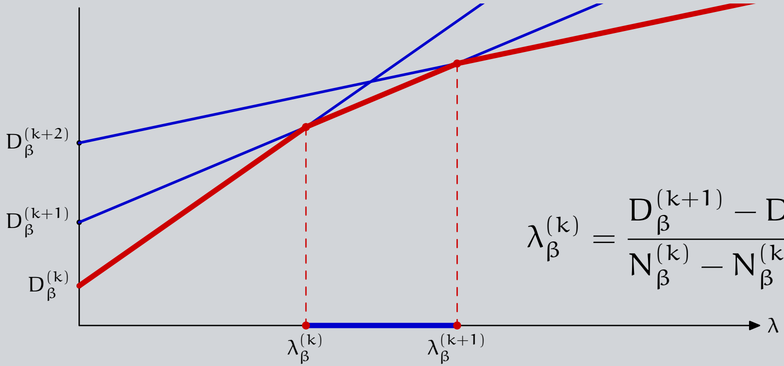


$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

Step 5: Optimal stategy for the Lagrange relaxation

Some inequalities $L_\beta^{(k)} < L_\beta^{(k+1)}, \quad M_\beta^{(k)} < M_\beta^{(k+1)}, \quad D_\beta^{(k)} < D_\beta^{(k+1)}.$

Lagrangian cost $C_\beta^{(k)}(\lambda) \coloneqq C(f^{(k)}, g^*; \lambda) = D_\beta^{(k)} + \lambda N_\beta^{(k)}$



$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

Optimal performance

- For all $\lambda \in (\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}]$ the threshold strategy $f^{(k+1)}$ is optimal.
- $C_\beta^*(\lambda) = \min_{k \in \mathbb{Z}} C_\beta^{(k)}$ is piecewise linear, continuous, concave, and increasing function of λ.

Step 6: Back to the constrained optimization problem

Bernoulli randomized strategy

Let $\theta \in [0, 1]$ and f_1 and f_2 be two stationary strategies.
The Bernoulli randomized strategy (f_1, f_2, θ) randomizes between f_1 and f_2 at each stage, choosing f_1 with probability θ and f_2 with probability $(1 - \theta)$.

Simple rand. strategy

A Bernoulli randomized strategy (f_1, f_2, θ) is simple if the actions prescribed by f_1 and f_2 differ only at one state.

Main result

Define $k^*_\beta = \sup\{k \in \mathbb{Z}_{\geqslant 0} : N^{(k)}_\beta \geqslant \alpha\}$ and let θ be such that

$$\theta N^{(k^*_\beta)}_\beta + (1 - \theta) N^{(k^*_\beta + 1)}_\beta = \alpha$$

Then, the Bernoulli simple randomized strategy $(f^{(k^*_\beta)}, f^{(k^*_\beta + 1)}, \theta)$ is optimal for the constrained optimization problem for $\beta \in (0, 1]$.

Step 6: Proof of the result

Sufficient condition for optimality

A (possibly randomized) strategy (f°, g°) is optimal for a constrained optimization problem with $\beta \in (0, 1]$ is the following conditions hold:

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$.

(C2) There exists a Lagrange multiplier $\lambda^\circ \geqslant 0$ such that (f°, g°) is optimal for $C_\beta(f^\circ, g^\circ; \lambda^\circ)$,

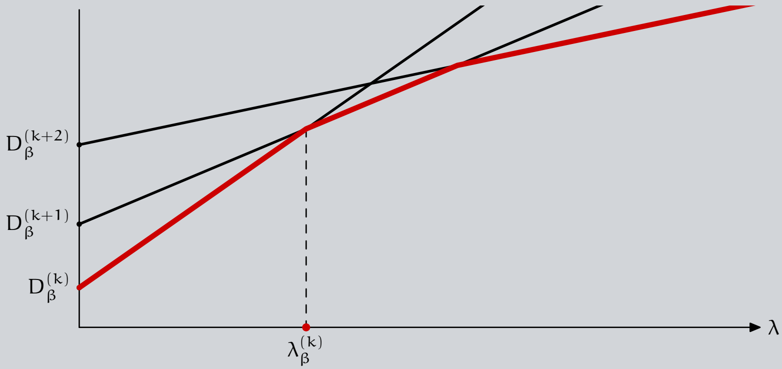
Step 6: Proof of the result

Sufficient condition for optimality

A (possibly randomized) strategy (f°, g°) is optimal for a constrained optimization problem with $\beta \in (0, 1]$ is the following conditions hold:

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$.

(C2) There exists a Lagrange multiplier $\lambda^\circ \geqslant 0$ such that (f°, g°) is optimal for $C_\beta(f^\circ, g^\circ; \lambda^\circ)$,



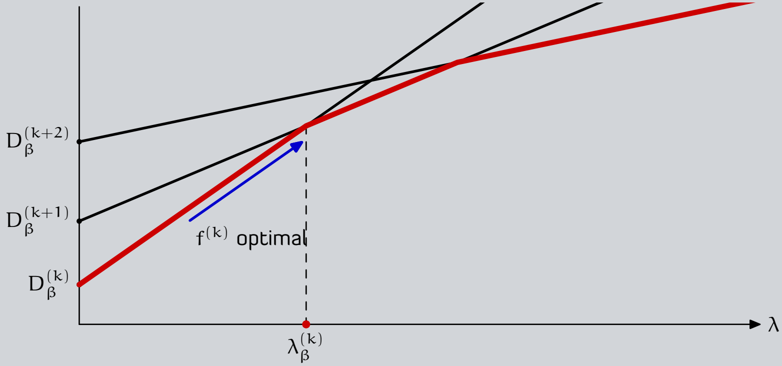
Step 6: Proof of the result

Sufficient condition for optimality

A (possibly randomized) strategy (f°, g°) is optimal for a constrained optimization problem with $\beta \in (0, 1]$ is the following conditions hold:

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$.

(C2) There exists a Lagrange multiplier $\lambda^\circ \geqslant 0$ such that (f°, g°) is optimal for $C_\beta(f^\circ, g^\circ; \lambda^\circ)$,



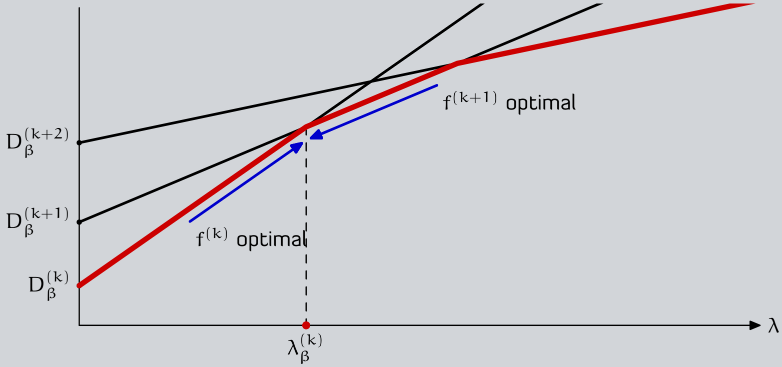
Step 6: Proof of the result

Sufficient condition for optimality

A (possibly randomized) strategy (f°, g°) is optimal for a constrained optimization problem with $\beta \in (0, 1]$ is the following conditions hold:

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$.

(C2) There exists a Lagrange multiplier $\lambda^\circ \geqslant 0$ such that (f°, g°) is optimal for $C_\beta(f^\circ, g^\circ; \lambda^\circ)$,

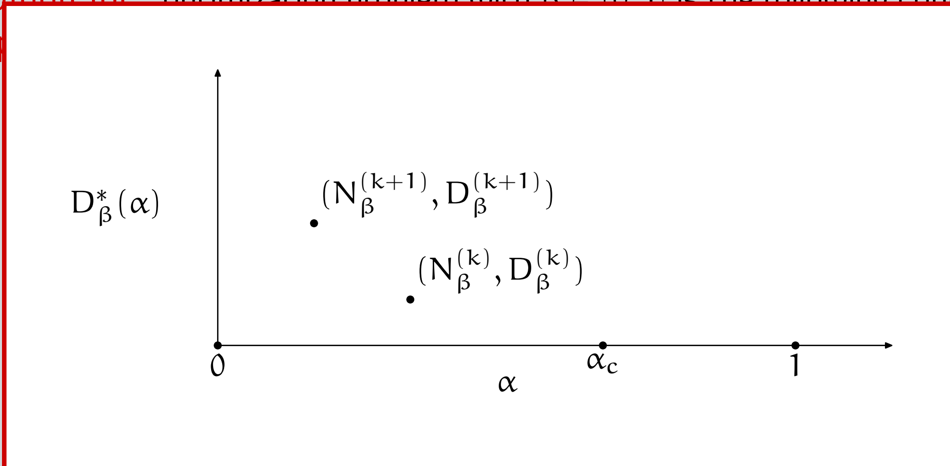


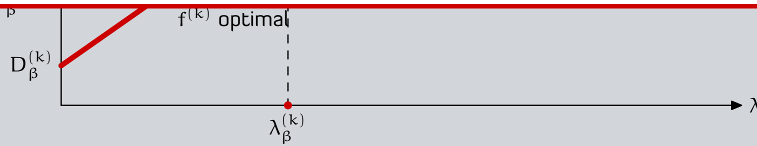
Step 6: Proof of the result

Sufficient condition for op…

A (possibly randomized) strategy (f°, g°) is optimal for a constrained optimization problem with $\beta \in (0, 1]$ is the following conditions hold:

…that (f°, g°) is



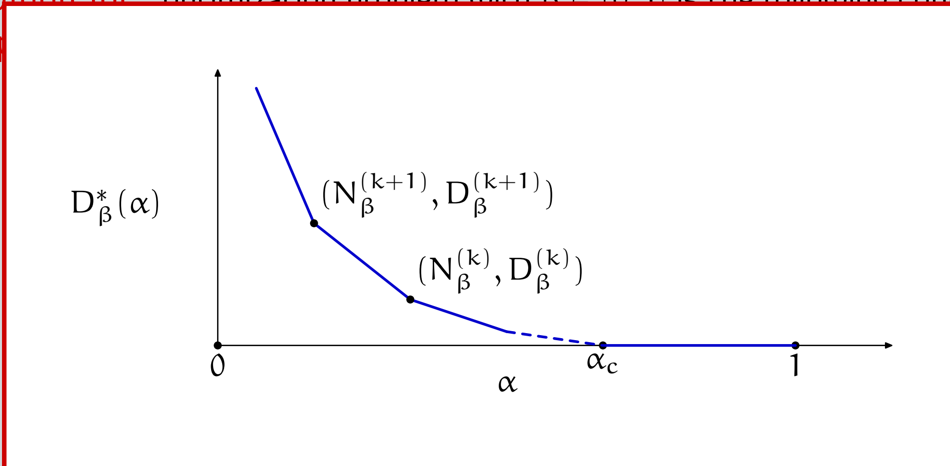


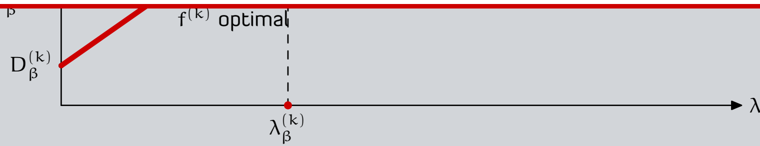
Step 6: Proof of the result

Sufficient condition for op

A (possibly randomized) strategy (f°, g°) is optimal for a constrained optimization problem with $\beta \in (0, 1]$ is the following conditions hold:

hat (f°, g°) is

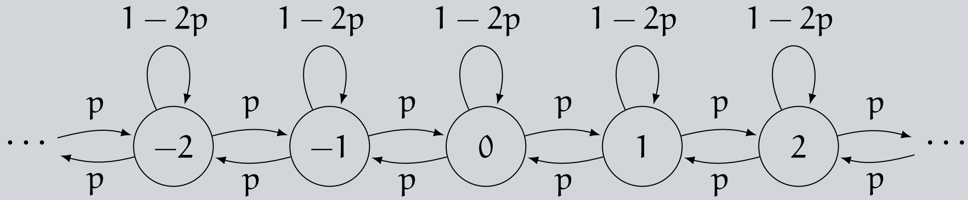




Completely characterize the distortion-transmission function and the optimal strategy that achieves any point on that function.

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$



An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

Discounted cost Let $K_\beta = -2 - (1-\beta)/\beta p$ and $m_\beta = \cosh^{-1}(-K_\beta/2)$.

$$D_\beta^{(k)} = \frac{\sinh(k m_\beta) - k \sinh(m_\beta)}{2 \sinh^2(k m_\beta/2) \sinh(m_\beta)}$$

$$N_\beta^{(k)} = \frac{2\beta p \sinh^2(m_\beta/2) \cosh(k m_\beta)}{\sinh^2(k m_\beta/2)} - (1-\beta)$$

Average cost $D_1^{(k)} = \dfrac{k^2-1}{3k}$ and $N_1^{(k)} = \dfrac{2p}{k^2}$

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where } p \in (0, \frac{1}{2}), \qquad d(e) = |e|$$

Discounted cost Let $K_\beta = -2 - (1-\beta)/\beta p$ and $m_\beta = \cosh^{-1}(-K_\beta/2)$.

$$D_\beta^{(k)} = \frac{\sinh(k m_\beta) - k \sinh(m_\beta)}{2\sinh^2(k m_\beta/2)\sinh(m_\beta)}$$

$$N_\beta^{(k)} = \frac{2\beta p \sinh^2(m_\beta/2)\cosh(k m_\beta)}{\sinh^2(k m_\beta/2)} - (1-\beta)$$

$\lambda_\beta^{(k)}$ can be computed in terms of $D_\beta^{(k)}$ and $N_\beta^{(k)}$.

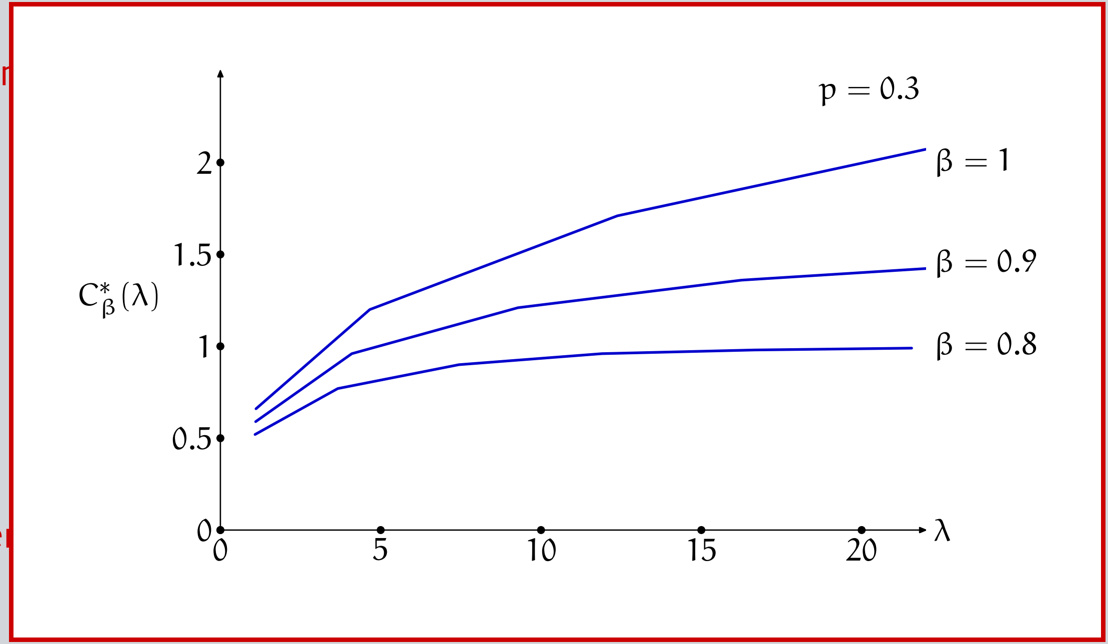
Average cost $D_1^{(k)} = \dfrac{k^2-1}{3k}$ and $N_1^{(k)} = \dfrac{2p}{k^2}$

$$\lambda_1^{(k)} = \frac{k(k+1)(k^2+k+1)}{6p(2k+1)}$$

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

Discour



Aver

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

Discounted cost Let $K_\beta = -2 - (1-\beta)/\beta p$ and $m_\beta = \cosh^{-1}(-K_\beta/2)$.

$$D_\beta^{(k)} = \frac{\sinh(k m_\beta) - k \sinh(m_\beta)}{2 \sinh^2(k m_\beta/2) \sinh(m_\beta)}$$

$$N_\beta^{(k)} = \frac{2\beta p \sinh^2(m_\beta/2) \cosh(k m_\beta)}{\sinh^2(k m_\beta/2)} - (1-\beta)$$

$$k_\beta^* = \sup\left\{ k \in \mathbb{Z}_{\geqslant 0} : \frac{\sinh^2(m_\beta/2)\cosh(k m_\beta)}{\sinh^2(k m_\beta/2)} \geqslant \frac{1+\alpha-\beta}{2\beta p} \right\}$$

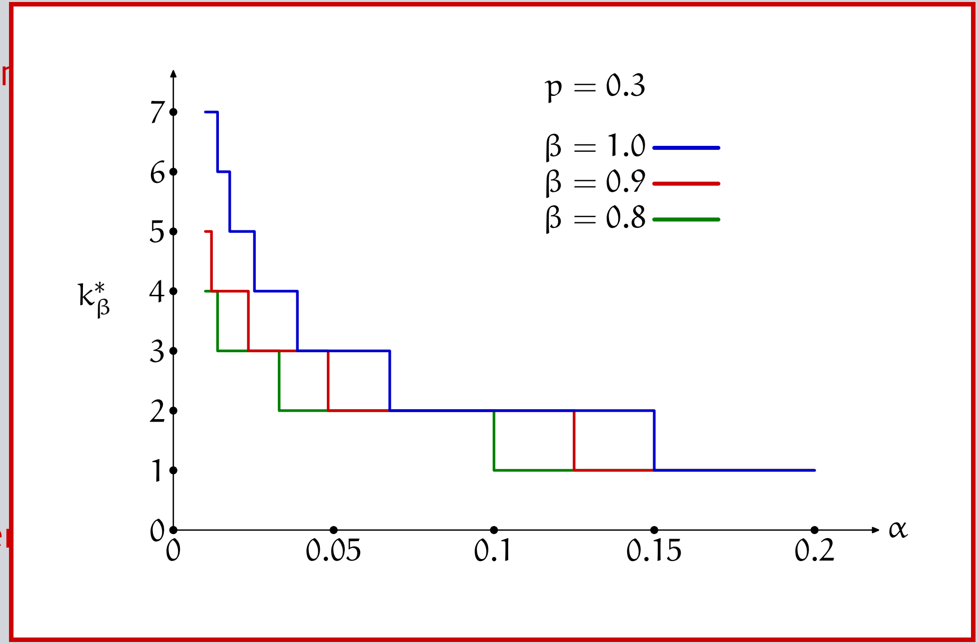
Average cost $D_1^{(k)} = \dfrac{k^2-1}{3k}$ and $N_1^{(k)} = \dfrac{2p}{k^2}$

$$k_1^* = \left\lfloor \sqrt{\frac{2p}{\alpha}} \right\rfloor$$

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

Discour

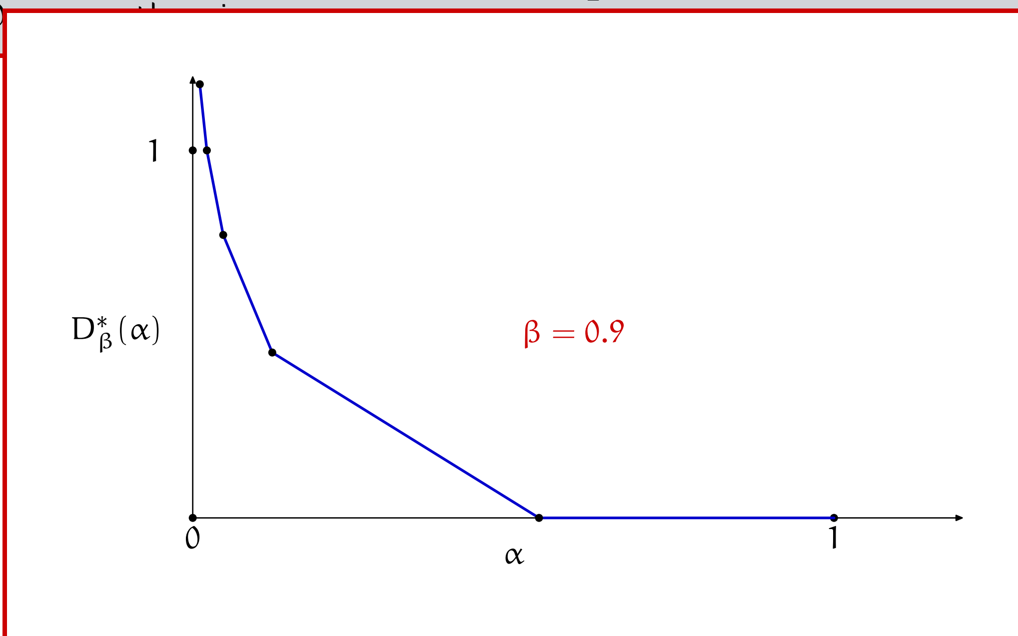


Aver

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0 \ldots \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

Discour…



Aver…

Summary and Conclusion

Problem formulation

- Real-time transmission of a Markov source under constraints on the number of transmissions.
- Investigated both discounted and average cost infinite horizon setups.
- Modeled as a decentralized stochastic control problem with two decision maker.
- As long as the transmitter uses a symmetric threshold based strategy, the estimation strategy does not depend on the transmission strategy.
- The problem of find the "best response" transmitter is a centralized stochastic control problem.

Main results

- Simple Bernoulli randomized strategies $(f^{(k^*)}, f^{(k^*+1)}, \theta)$ are optimal.
- k^* and θ can be computed easily.
- Characterize the distortion-transmission function

References

- Chakravorty and Mahajan, Allerton 2014.
- Chakravorty and Mahajan, CDC 2014
- Full paper to be posted to arxiv soon.