Remote estimation of Markov processes under communication constraints

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Joint work with Jhelum Chakravorty

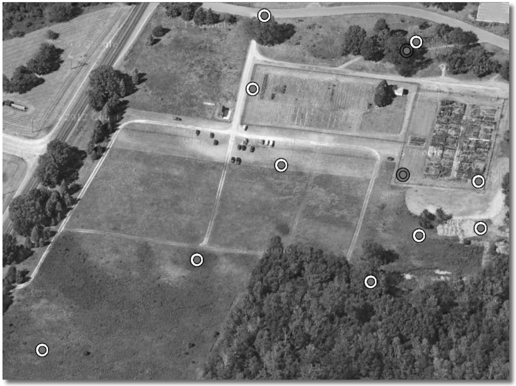
Applied Probability Conference
Rutgers University, 2–3 Oct 2015

Motivation

Many applications require:

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction

Motivation



Sensor Networks

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- Sequential transmission of data
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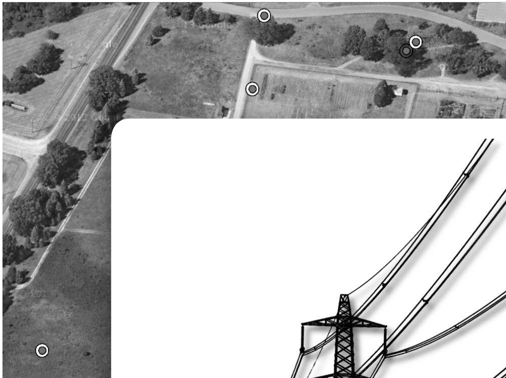

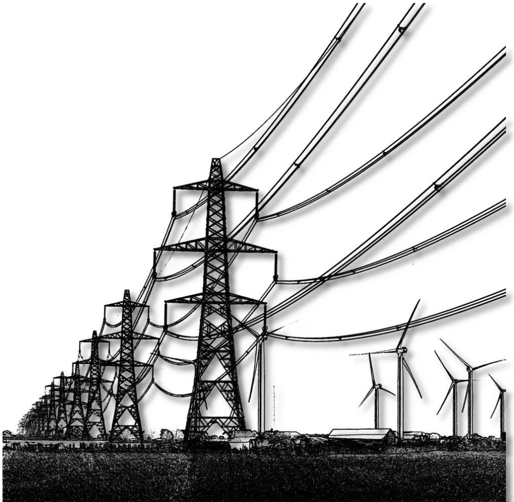

Smart Grids

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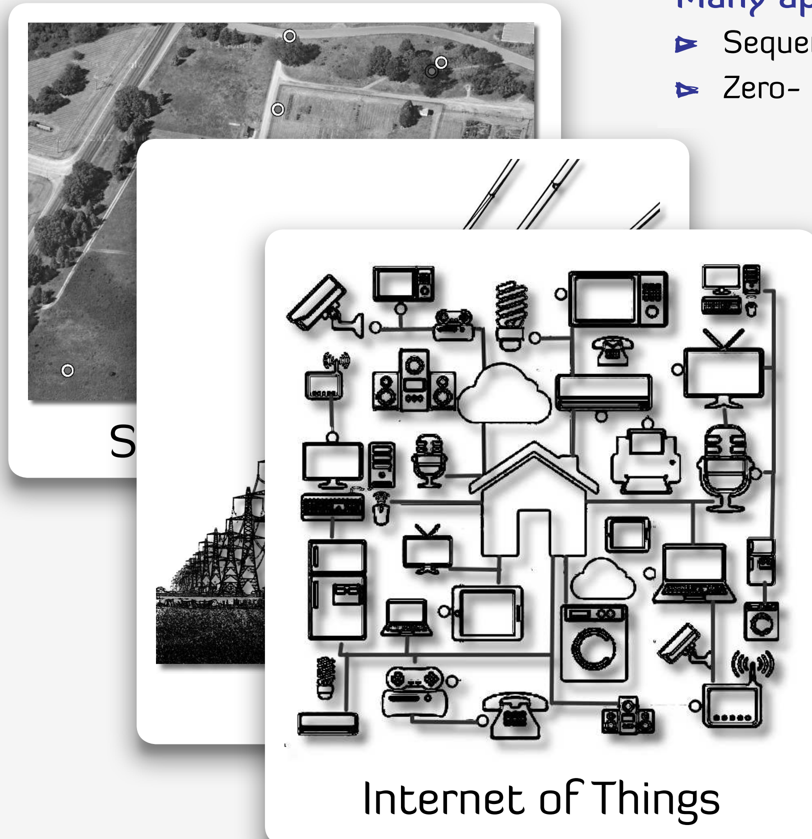
Motivation



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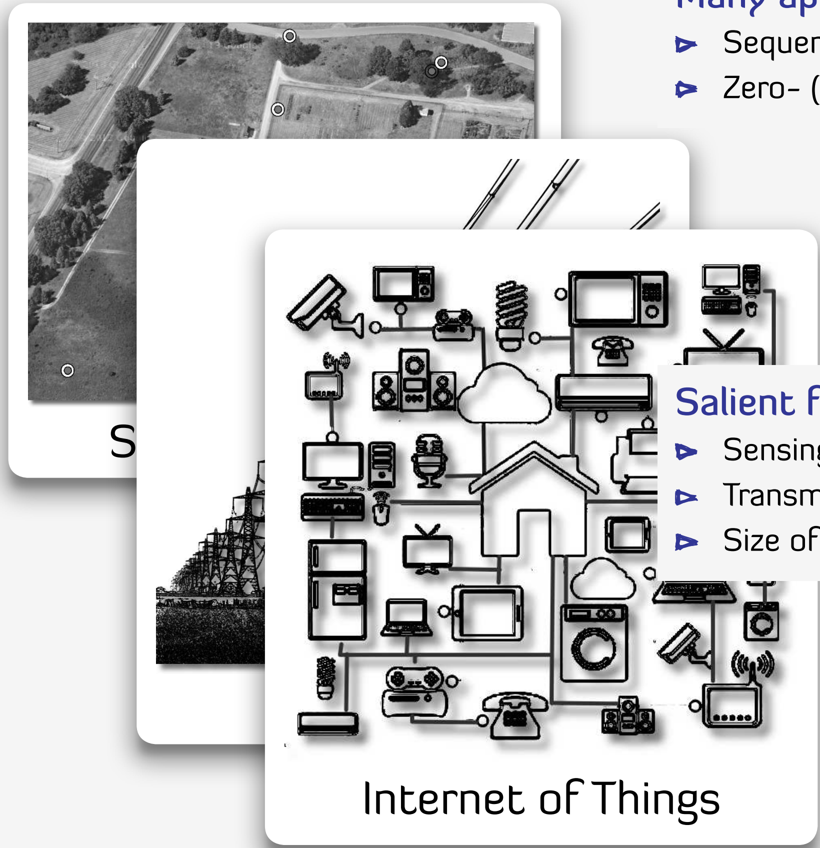

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- Zero- (or finite-) delay reconstruction

Salient features

- Sensing is cheap
- Transmission is expensive
- Size of data-packet is not critical

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Analyze a stylized model and evaluate fundamental trade-offs

A completely solved example of a "simple" decentralized system with non-classical information structure

Brief overview of decentralized stochastic control

Economics Literature

- Marschak, “Elements for a Theory of Teams,” Management Science, 1955
- Radner, ”Team decision problems,” Ann Math Stat, 1962.
- Marschak and Radner, “Economics Theory of Teams,” 1972.
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Systems and Control Literature

- Witsenhausen, “Separation of estimation and control,” Proc IEEE, 1971.
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Artificial Intelligence Literature

- . . .

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Simpler than non-cooperative game theory.

All “pre-game” agreements are enforceable.

Simpler than cooperative game theory.

The value of the game does not need to be split between the players.

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Main difficulty: Seeking global optimality

Simpl

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Conceptual difficulties in decentralized control

Witsenhausen Counterexample

- A two step dynamical system with two controllers
- Linear dynamics, quadratic cost, and Gaussian disturbance
- Non-linear controllers outperform linear control strategies . . .

  . . . cannot use Kalman filtering + Riccati equations

Whittle and Rudge Example

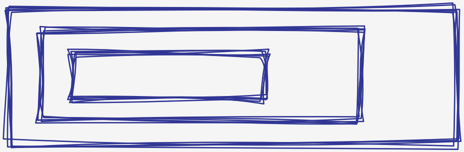
- Infinite horizon dynamical system with two symmetric controller
- Linear dynamics, quadratic cost, and Gaussian disturbance
- A priori restrict attention to linear controllers
- Best linear controllers not representable by recursions of finite order

Complexity analysis

- All random variables are finite valued
- Finite horizon setup
- The problem of finding the best control strategy is in NEXP

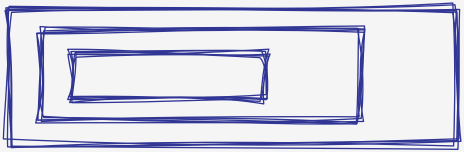
- Witsenhausen, "A counterexample in stochastic optimum control," SICON 1969.
- Whittle and Rudge, "The optimal linear solution of a symmetric team control problem," App. Prob. 1974.
- Bernstein, et al, "The complexity of decentralized control of Markov decision processes," MOR 2002.
- Goldmand and Zilberstein, "Decentralized control of cooperative systems: categorization and complexity," JAIR 2004.

Solution concepts



Classical info. struct.

Solution concepts



Classical info. struct.

- Structure of optimal strategies
  Instead of f(history of obs) use f(info state).

- Compute optimal strategy using DP
  $$V(\text{info state}) = \min_{\text{action}} \left[\mathcal{B}_{\text{action}} V\right](\text{info state})$$

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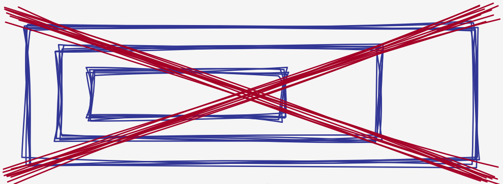

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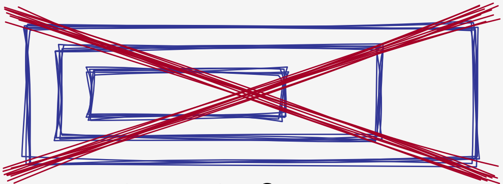

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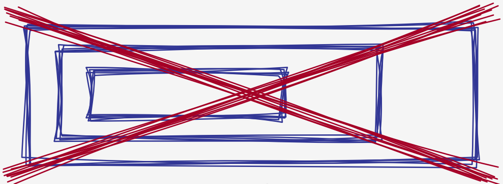

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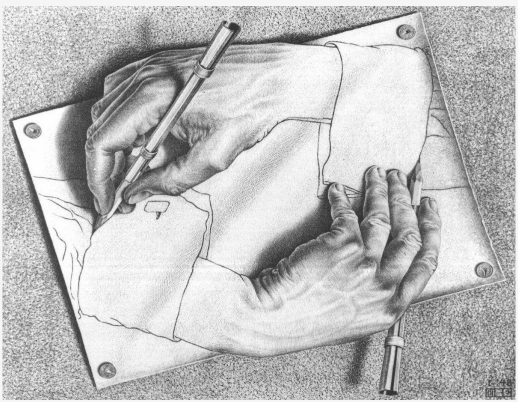

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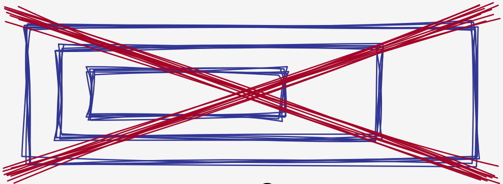

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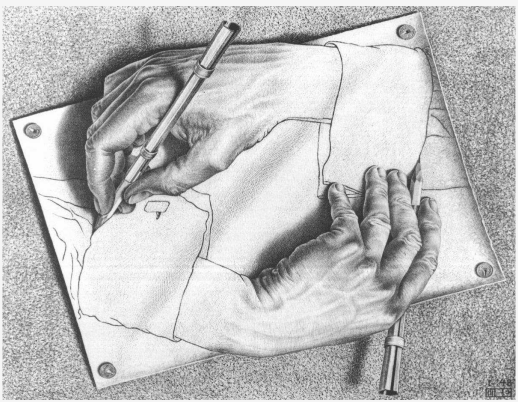

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Common-information approach

- Structure of optimal strategies
  Instead of $f(\text{history of obs})$
  use $f(\text{local info}, \text{common info based state})$.
- Compute optimal strategy using DP
  $V(\text{info state}) = \min_{\varphi:\text{local data} \to \text{action}} \big[\mathcal{B}_{\varphi} V\big](\text{info state})$

Nayyar, Mahajan, Teneketzis, “Decentralized stochastic control with partial history sharing”, TAC 2013.

Solution concepts

Allows us to use tools from MDP literature
to decentralized stochastic control

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No general solution methodology

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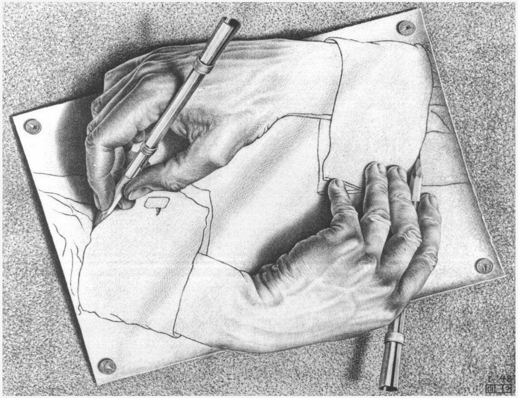

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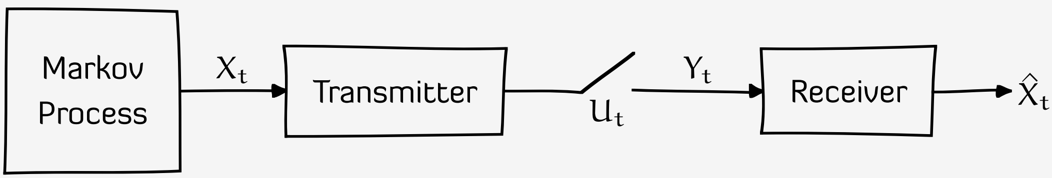
The system model



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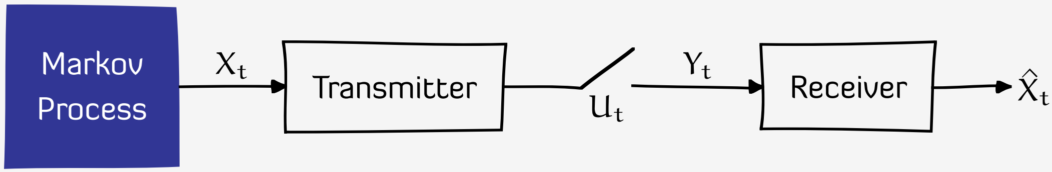
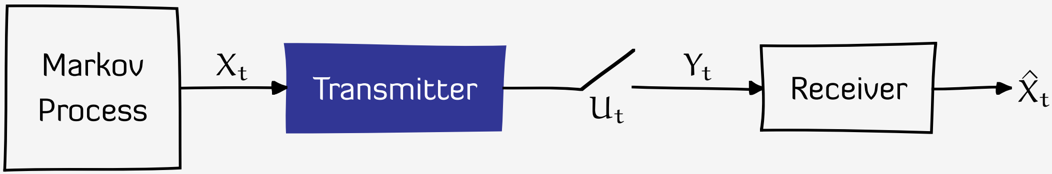
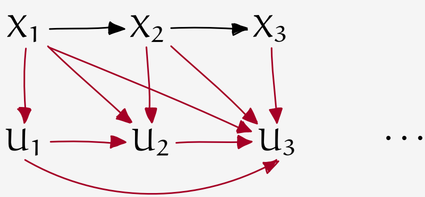


The system model

$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \varepsilon, & \text{if } U_t = 0 \end{cases}$$



$$U_t = f_t(X_{1:t}, U_{1:t-1})$$



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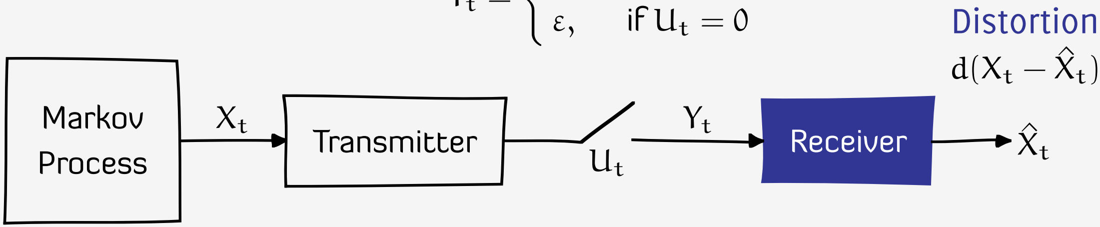
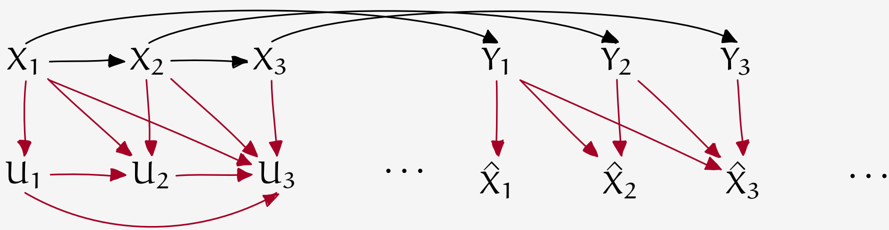


$U_t = f_t(X_{1:t}, U_{1:t-1})$

$\hat{X}_t = g_t(Y_{1:t})$



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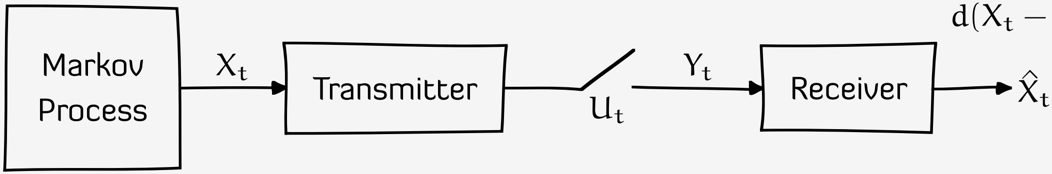
Distortion

$d(X_t - \hat{X}_t)$



$$U_t = f_t(X_{1:t}, U_{1:t-1}) \qquad \hat{X}_t = g_t(Y_{1:t})$$

Communication Strategies

- Transmission strategy $f = \{f_t\}_{t=0}^{\infty}$.
- Estimation strategy $g = \{g_t\}_{t=0}^{\infty}$.

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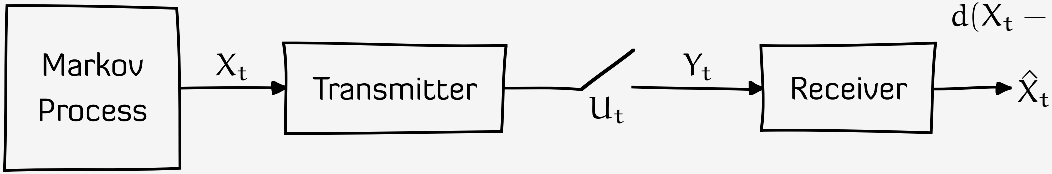

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1. Discounted setup, $\beta \in (0, 1)$

$$D_\beta(f, g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t)\Big]; \qquad N_\beta(f, g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t\Big]$$

2. Average cost setup, $\beta = 1$

$$D_1(f, g) = \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t)\Big]; \qquad N_1(f, g) = \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} U_t\Big]$$

Optimization problems

Costly communication

$$\text{For } \lambda \in \mathbb{R}_{>0}, \quad C^*_\beta(\lambda) = C_\beta(f^*, g^*; \lambda) \coloneqq \inf_{(f,g)} \left\{ D_\beta(f,g) + \lambda N_\beta(f,g) \right\}$$

Constrained communication

$$\text{For } \alpha \in (0,1), \quad D^*_\beta(\alpha) \coloneqq \inf_{(f,g)} \left\{ D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha \right\}$$

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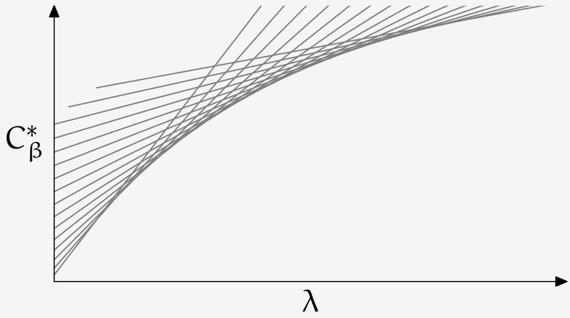

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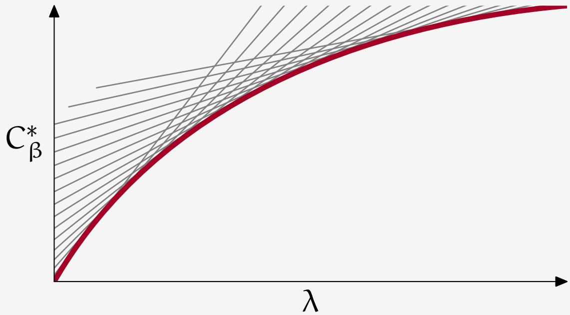

C^*_β is cts, inc, and concave

Optimization problems

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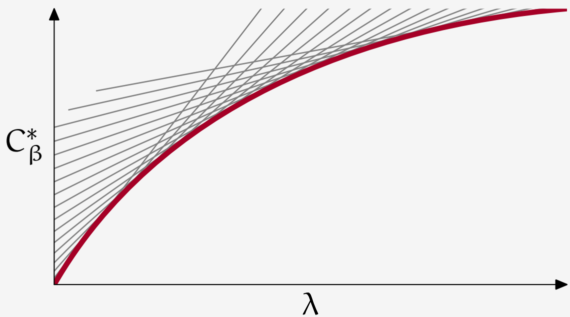

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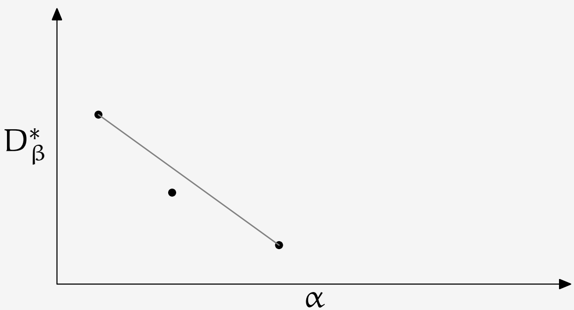

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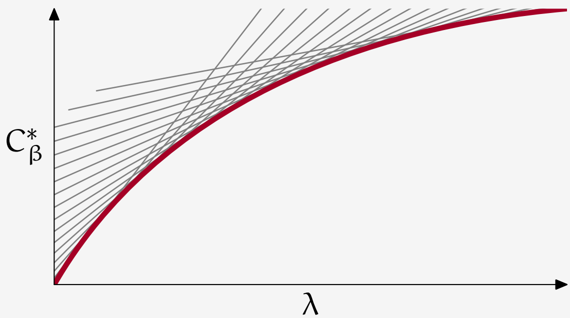

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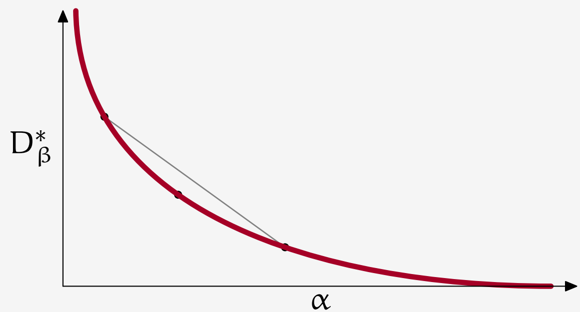


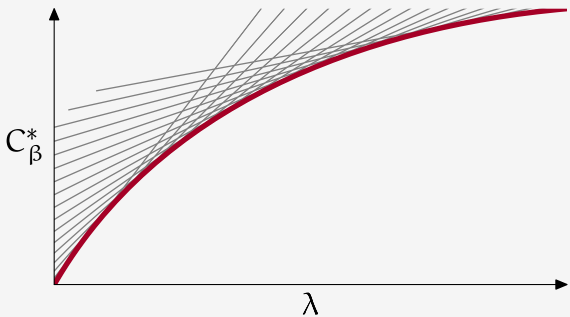
D^*_β is cts, dec, and convex

Optimization problems

Costly communication

$\lambda N_\beta(f, g)\}$

Our result: Provide computable expressions for these curves and identify strategies that achieve them.



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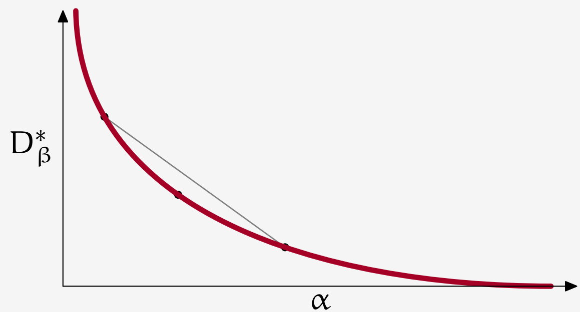

D^*_β is cts, dec, and convex

$$X_{t+1} = X_t + W_t,\ W_t \sim \mathcal{N}(0, 1)$$

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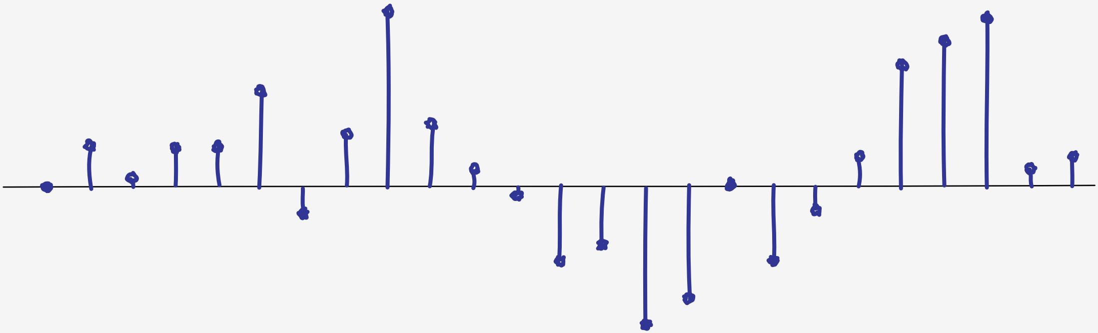
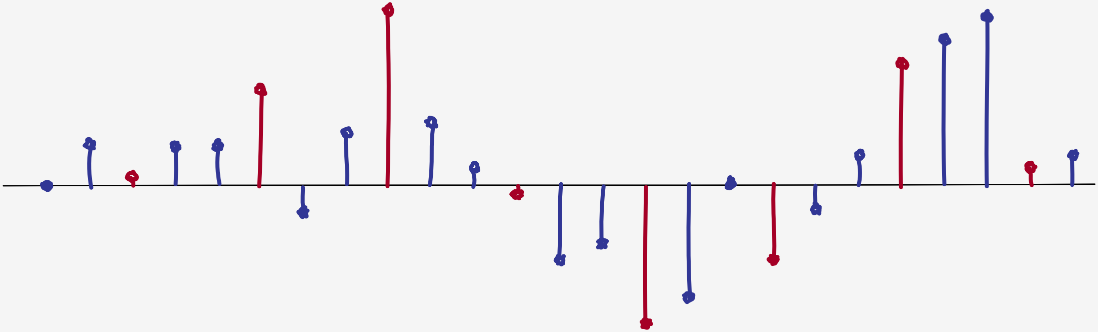


Periodic transmission strategy



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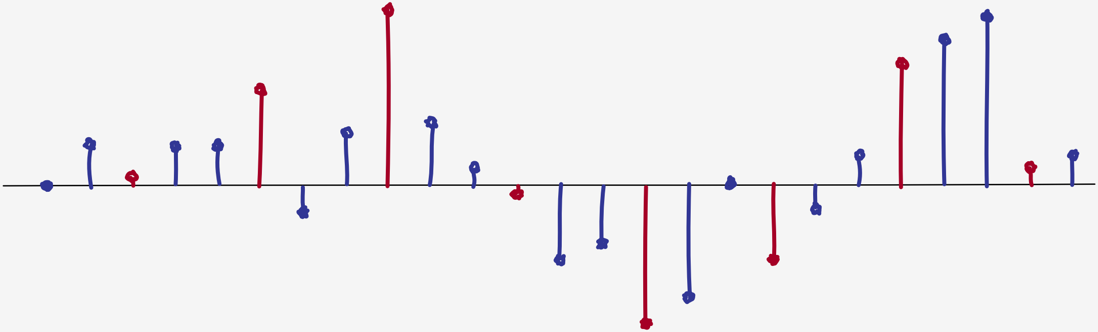
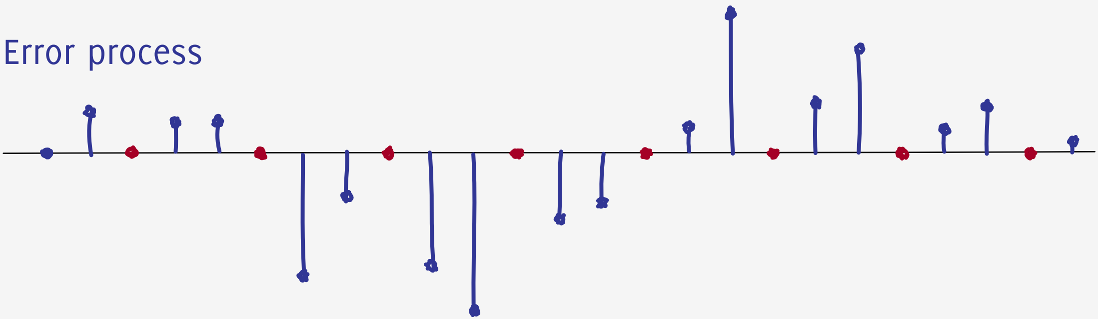


Error process



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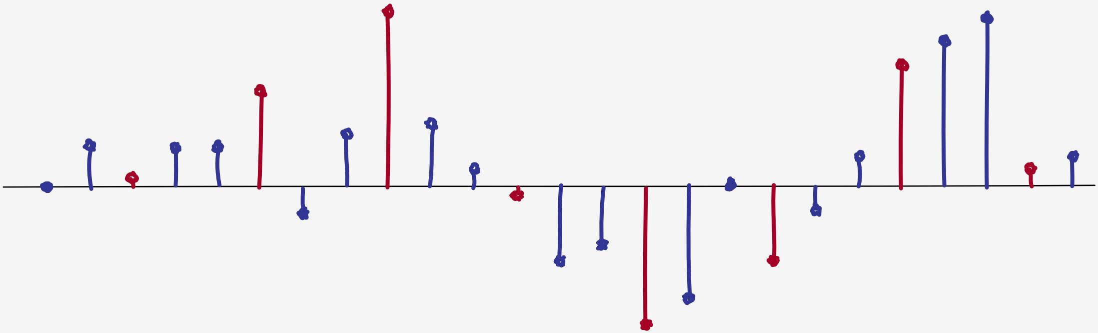

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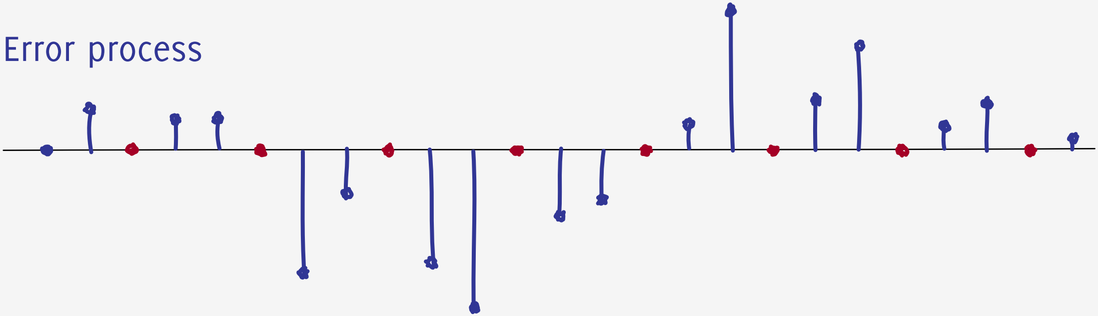
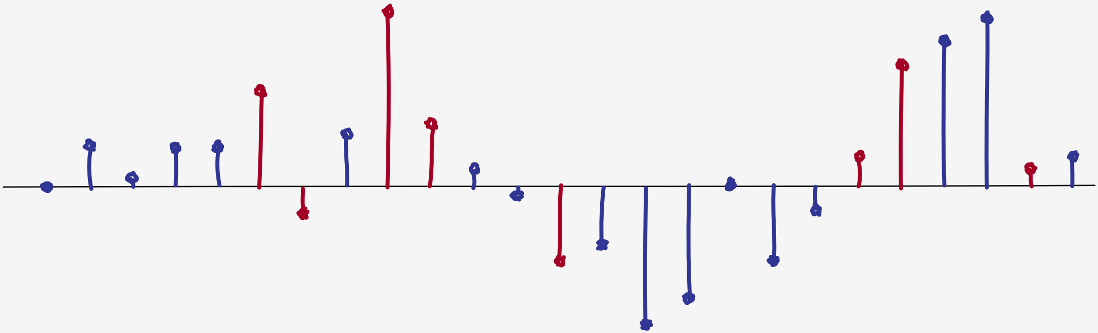


$D = 0.69 \quad N \approx 1/3$

An alternative strategy



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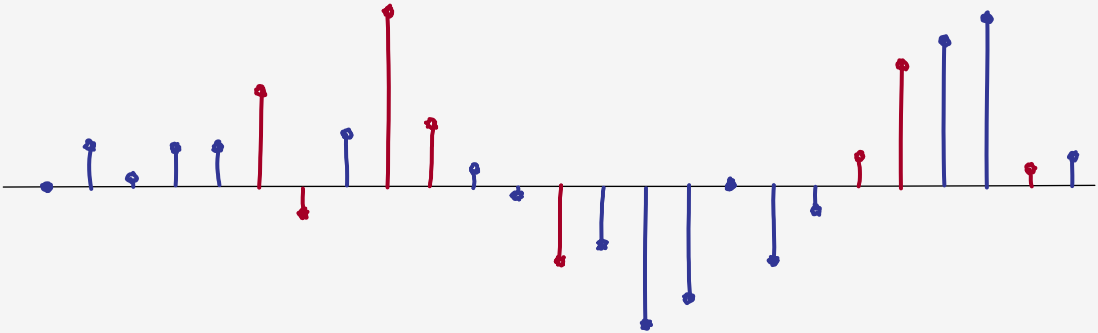

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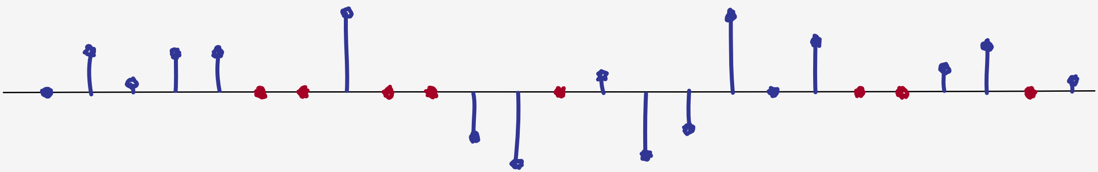

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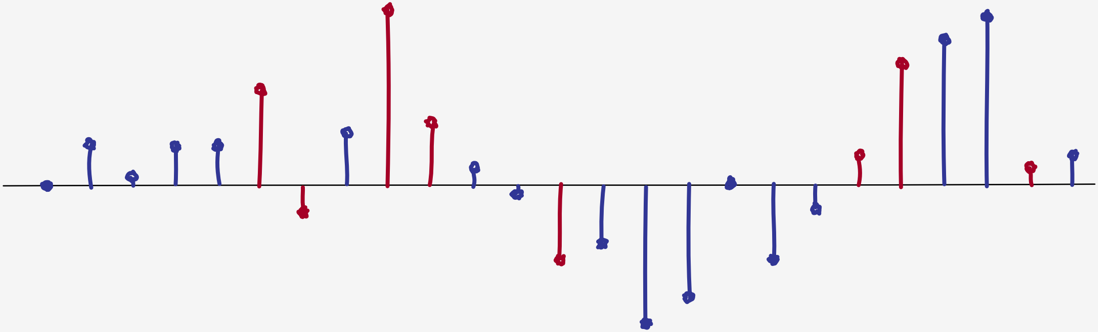

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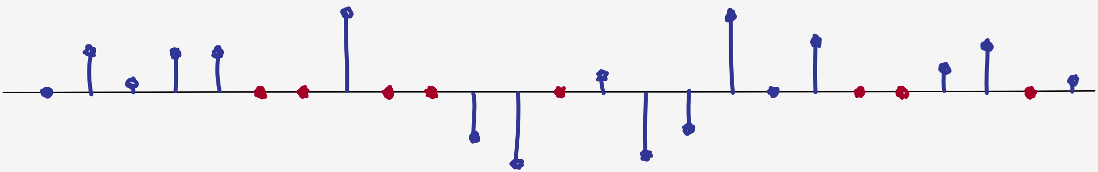
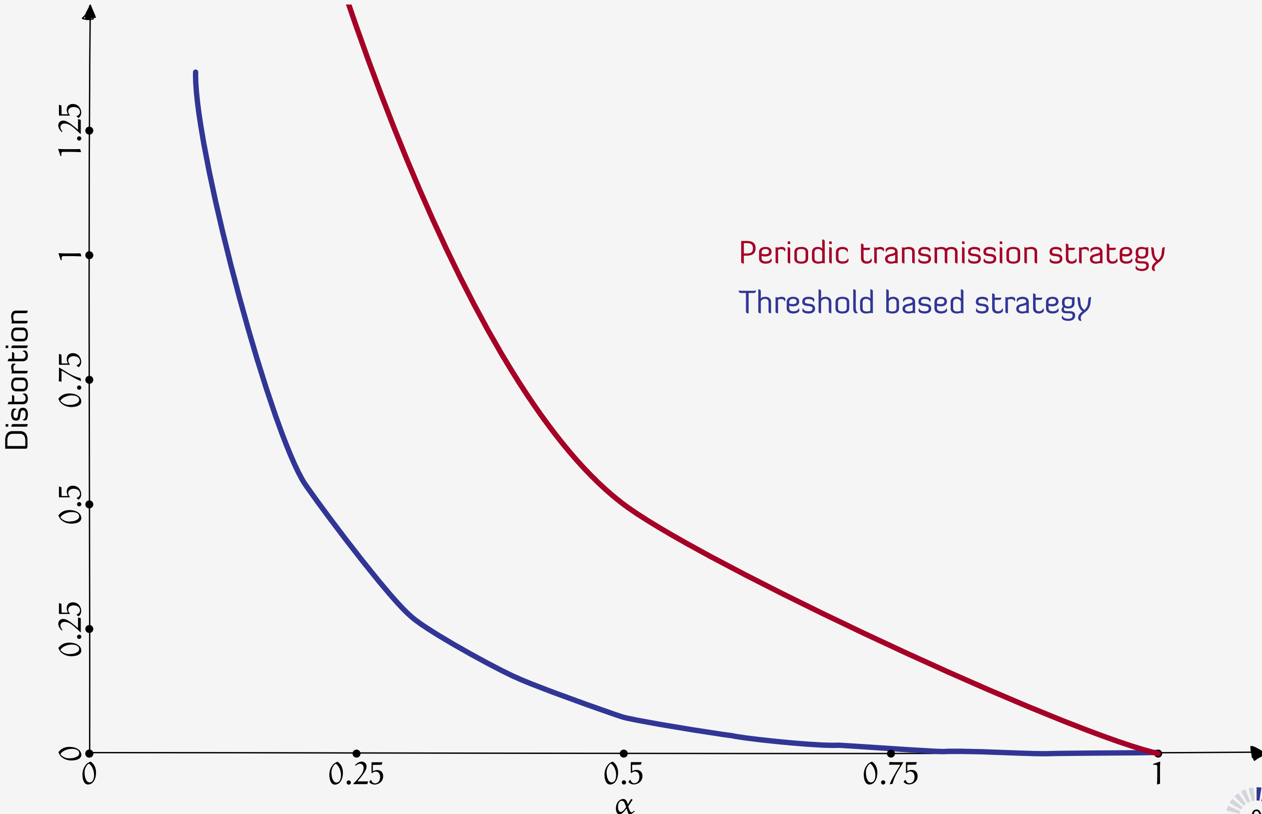


$D = 0.24 \quad N \approx 1/3$

Distortion-transmission function



Identify strategies that achieve the optimal trade-off

Simple and intuitive threshold based strategies are optimal.

Provide computable expressions for distortion-transmission function

Based on simple matrix calculations for discrete Markov processes

Based on solving Fredholm integral equations for Gaussian processes

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Beautiful example of stochastics and optimization

Decentralized stochastic control and POMDPs

Stochastic orders and majorization

Markov chain analysis, stopping times, and renewal theory

Constrained MDPs and Lagrangian relaxations

So how do we start?

Decentralized stochastic control

The common information approach

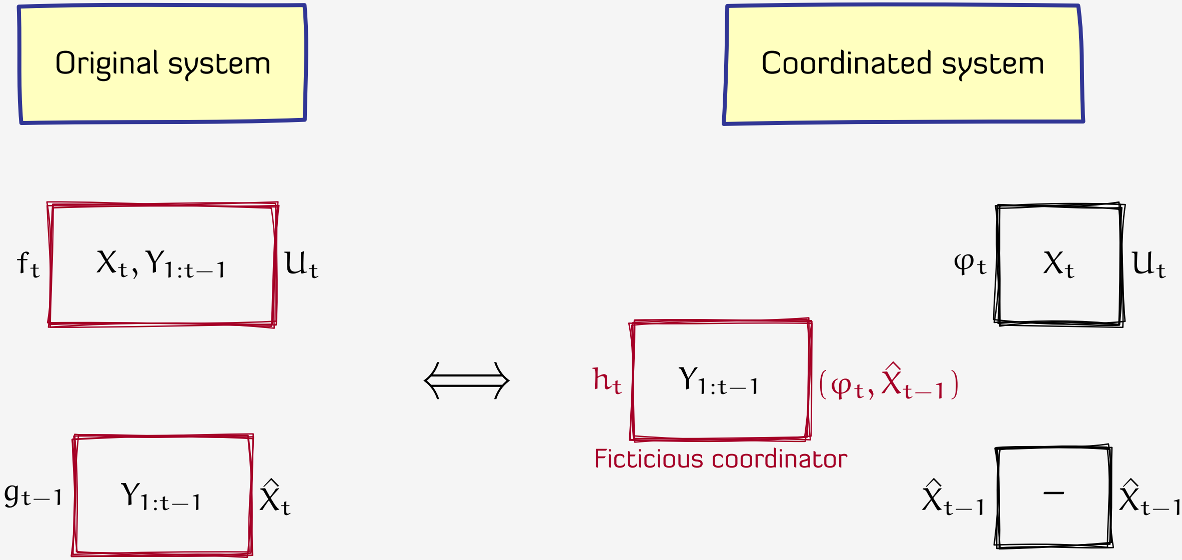
Original system

f_t | $X_t, Y_{1:t-1}$ | U_t

g_{t-1} | $Y_{1:t-1}$ | $\hat{X}_t$

Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

The common information approach



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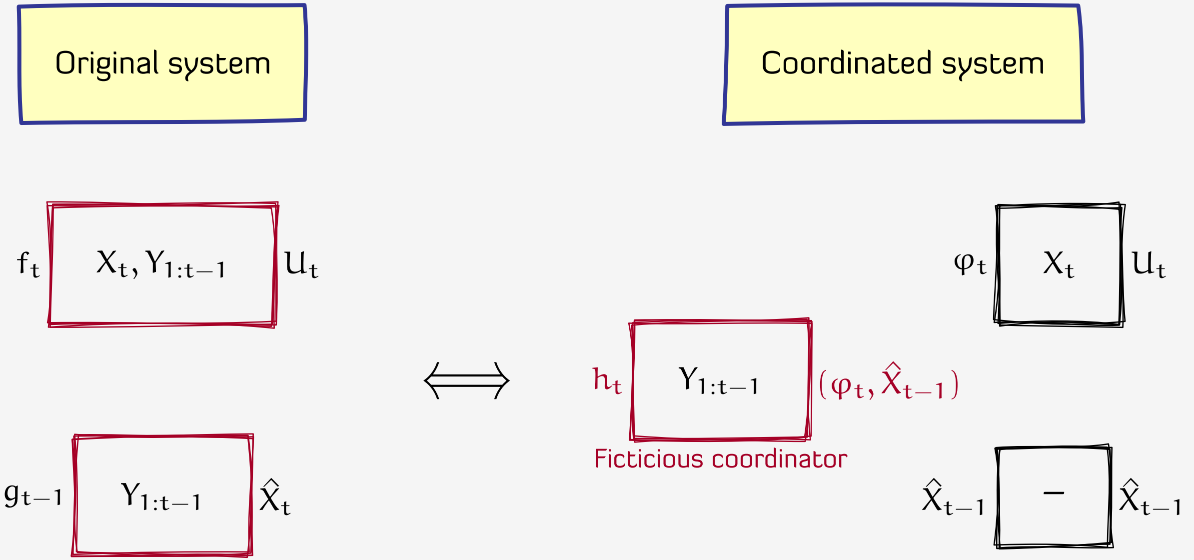

- The coordinated system is equivalent to the original system.

$$f_t(x, y_{1:t-1}) = h_t^1(y_{1:t-1})(x).$$

- The coordinated system is centralized. Belief state $\mathbb{P}(X_t \mid Y_{1:t-1})$.

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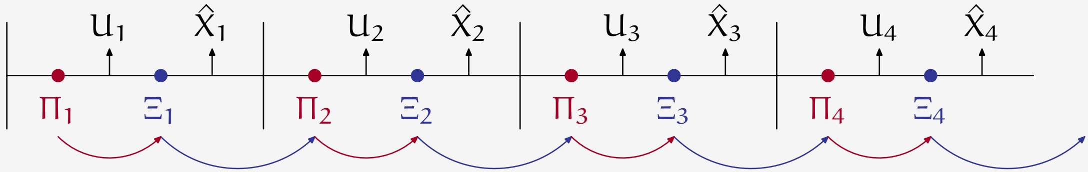
Information states and dynamic program

Information states

Pre-transmission belief : $\Pi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t-1})$.

Post-transmission belief : $\Xi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t})$.



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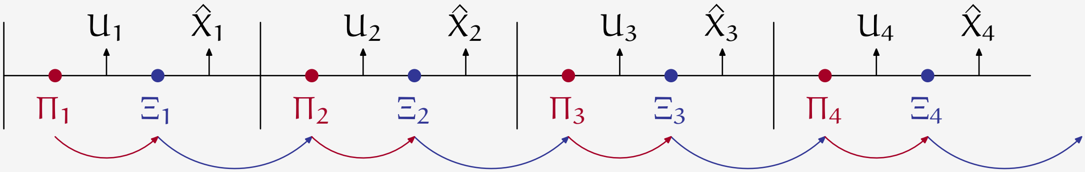

Structural results

There is no loss of optimality in using

$$U_t = f_t(X_t, \Pi_t) \quad \text{and} \quad \hat{X}_t = g_t(\Xi_t).$$

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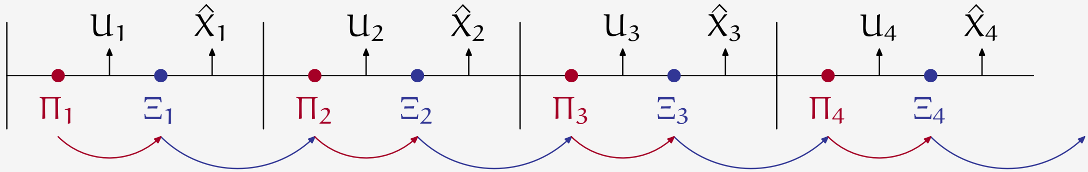

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Dynamic Program

$$W_{T+1}(\pi) = 0$$

and for $t = T, \dots, 0$

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Information states and dynamic program

Informa … $x \mid Y_{1:t-1})$.

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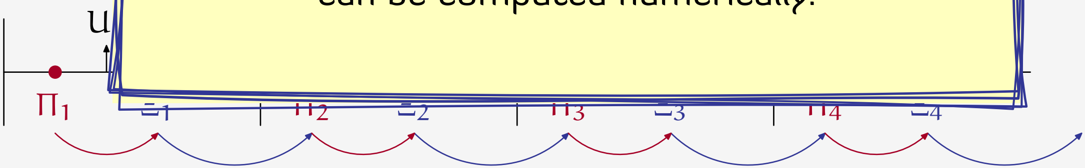
"Standard" POMDP. Optimal strategies can be computed numerically.



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Can we use the DP to say something more about the optimal strategy?

Simplifying modeling assumptions

Markov process $X_{t+1} = aX_t + W_t$

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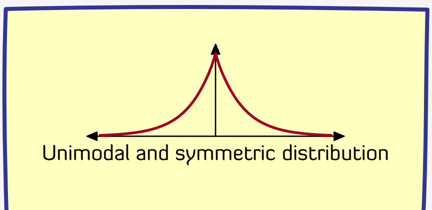
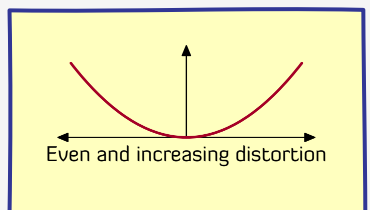
	Markov chain setup	**Guass-Markov setup**
State spaces	$X_t, a, W_t \in \mathbb{Z}$	$X_t, a, W_t \in \mathbb{R}$
Noise distribution	Unimodal and symmetric $p_e = p_{-e} \geqslant p_{e+1}$	Zero-mean Gaussian $\varphi_\sigma(\cdot)$
Distortion	Even and increasing $d(e) = d(-e) \leqslant d(e+1)$	Mean-squared $d(e) = \|e\|^2$





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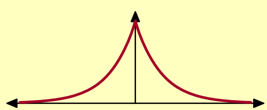
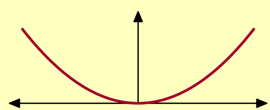

Unimodal and symmetric distribution


Even and increasing distortion

Step 1 Structure of optimal strategies

Step 2 Performance of arbitrary threshold strategies $f^{(k)}$

Step 3 Optimal costly comm.

Step 4 Distortion-transmission trade-off

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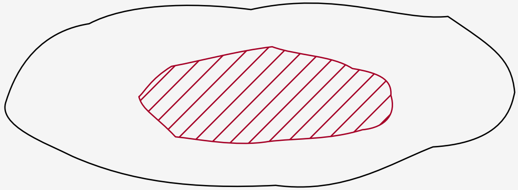

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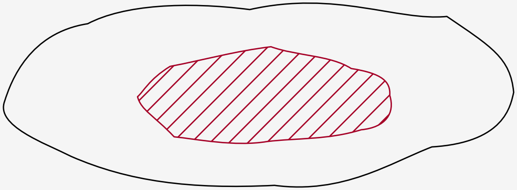

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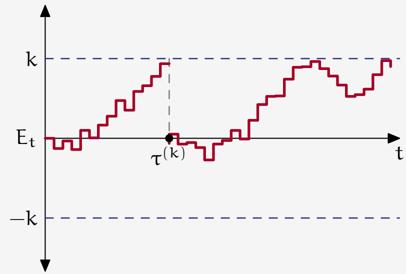

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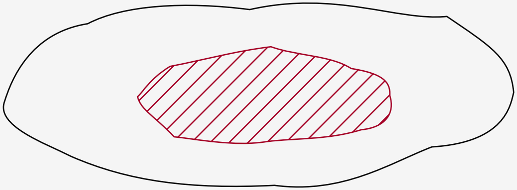

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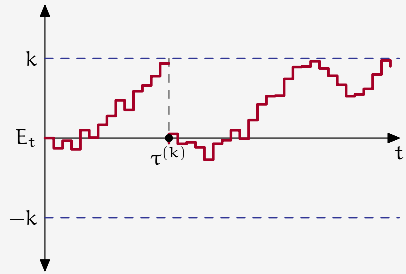

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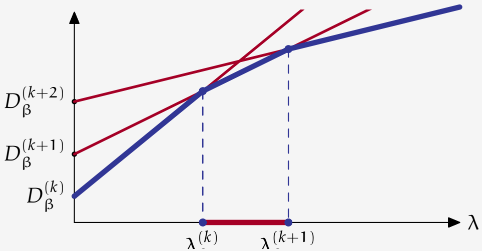

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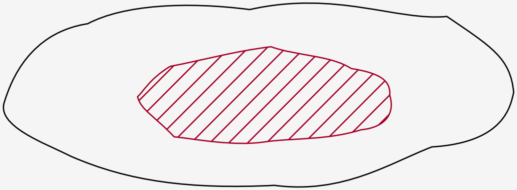

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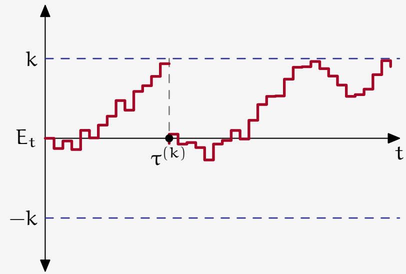

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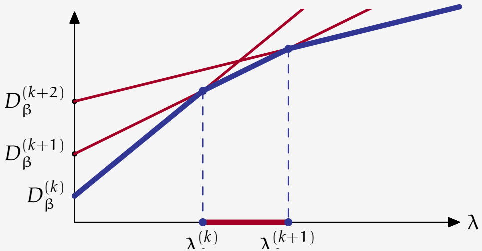

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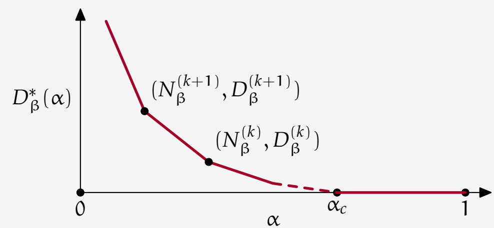

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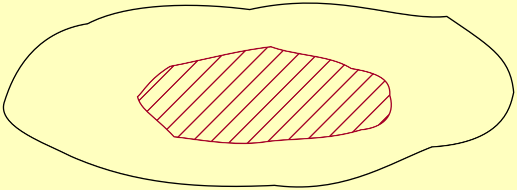

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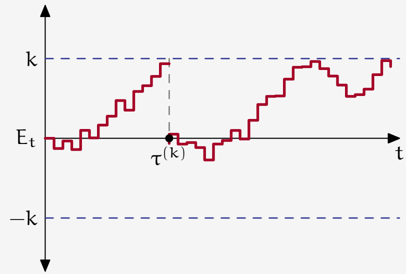

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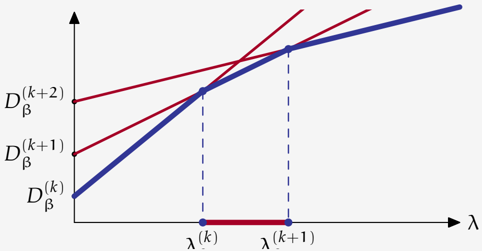

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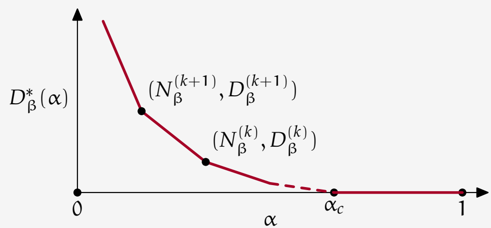

Step 1 Structure of optimal strategies (finite horizon)

Oblivious estimation process

$$Z_t = \begin{cases} X_t & \text{if } U_t = 1 \text{ (or } Y_t \neq \varepsilon) \\ aZ_{t-1} & \text{if } U_t = 0 \text{ (or } Y_t = \varepsilon) \end{cases}$$

Error process

$$E_t = X_t - aZ_{t-1}$$

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Error process

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Optimal estimator

$$\hat{X}_t = g_t^*(Z_t) = Z_t$$

Optimal transmitter

There exists thresholds $\{k_t\}_{t=0}^{\infty}$ such that

$$U_t = f_t^*(E_t) = \begin{cases} 1 & \text{if } |E_t| \geqslant k_t \\ 0 & \text{if } |E_t| < k_t \end{cases}$$

Some comments

The result is non-intuitive

- The transmitter does not try to send information through timing information.
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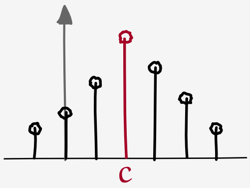
Proof outline

- . . .

Preliminaries

[LM11, NBTV13]

Almost uniform and unimodal (ASU) distribution about c



$\pi_c \geqslant \pi_{c+1} \geqslant \pi_{c-1} \geqslant \pi_{c+2} \geqslant \cdots$

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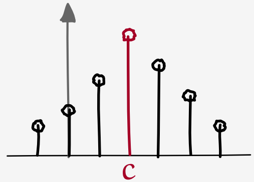

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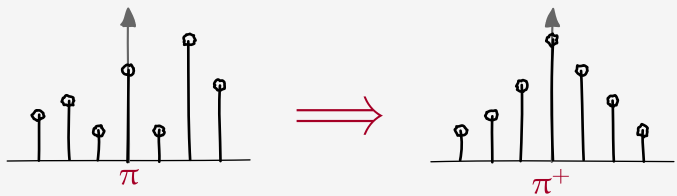
ASU Rearrangement



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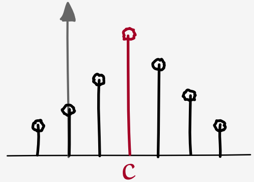

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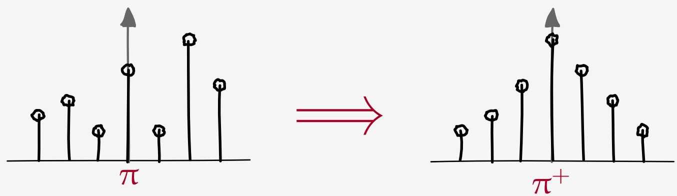


Majorization

$\pi \succeq \xi$ iff

$$\sum_{i=-n}^{n} \pi_i^+ \geqslant \sum_{i=-n}^{n} \xi_i^+ \quad \text{and} \quad \sum_{i=-n}^{n+1} \pi_i^+ \geqslant \sum_{i=-n}^{n+1} \xi_i^+$$

Invariant to permutations.

Proof outline

[LM11, NBTV13]

Use backward induction to show that value function is "almost" Schur-concave

- If $\xi' \succeq \xi$ and ξ is ASU, then $V_t(\xi') \geqslant V_t(\xi)$
- If $\pi' \succeq \pi$ and π is ASU, then $W_t(\pi') \geqslant W_t(\pi)$

Use backward induction to show that

- If ξ is ASU about c, then c is the arg min of

$$V_t(\xi) = \min_{\hat{x} \in \mathcal{X}} \mathbb{E}[d(X_t - \hat{x}) + W_{t+1}(\Pi_{t+1}) \mid \Xi_t = \xi],$$

- If π is ASU about c, then the arg min of

$$W_t(\pi) = \min_{\varphi : \mathcal{X} \to \{0,1\}} \mathbb{E}[\lambda \varphi(X_t) + V_t(\Xi_t) \mid \Pi_t = \pi, \varphi_t = \varphi]$$

  is of the threshold form in $|x - ac|$.

Use forward induction to show that under the optimal strategy

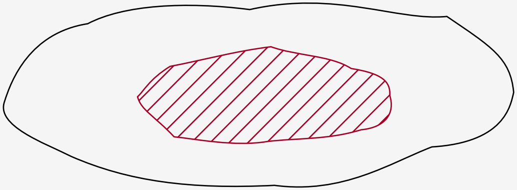
- Π_t is ASU around Z_{t-1}
- Ξ_t is ASU around Z_t

The results extend to infinite horizon setup under appropriate regularity conditions.

Time-homogeneous threshold-based strategies are optimal.

How do we find the optimal threshold-based strategy?

Step 1 Structure of optimal strategies



Search space of strategies (f, g)

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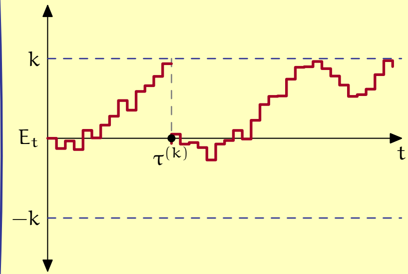

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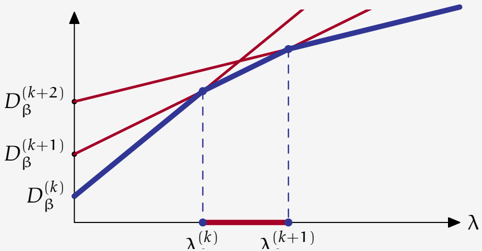

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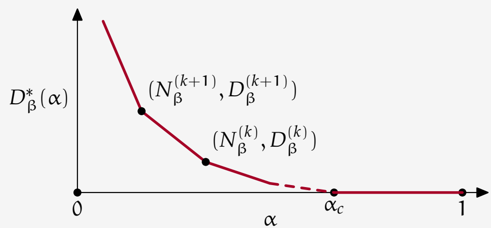
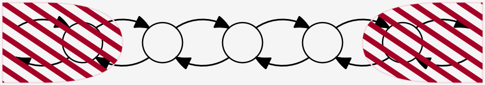


Step 2 Performance of threshold strategies

Consider a threshold-based strategy

$$f^{(k)}(e) = \begin{cases} 1 & \text{if } |e| \geqslant k \\ 0 & \text{otherwise} \end{cases}$$



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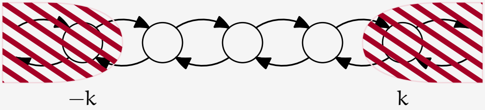
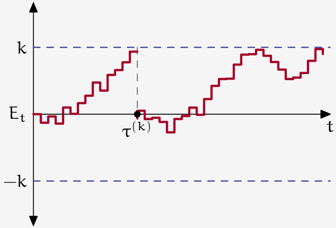


Let $\tau^{(k)}$ denote the stopping time of first transmission (starting at $E_0 = 0$).



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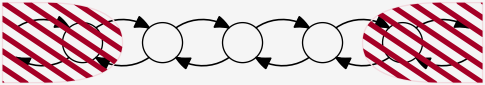

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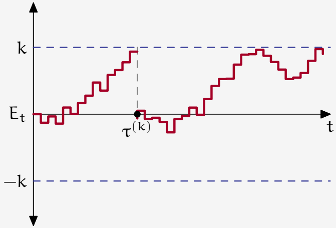

Define

$$L_\beta^{(k)}(e) = (1-\beta)\,\mathbb{E}\left[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \,\middle|\, E_0 = e\right].$$

$$M_\beta^{(k)}(e) = (1-\beta)\,\mathbb{E}\left[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \,\middle|\, E_0 = e\right].$$

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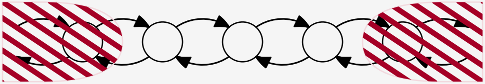

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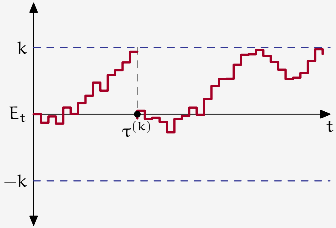

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Proposition $\{E_t\}_{t=0}^{\infty}$ is a regenerative process. By renewal theory,

$$D_\beta^{(k)} \coloneqq D_\beta(f^{(k)}, g^*) = \frac{L_\beta^{(k)}(0)}{M_\beta^{(k)}(0)} \quad \text{and} \quad N_\beta^{(k)} \coloneqq N_\beta(f^{(k)}, g^*) = \frac{1}{M_\beta^{(k)}(0)} - (1-\beta).$$

Step 2 Performance of threshold strategies

Conside... of

...0)).

$f^{(k)}$

$-k$

Computing $L^{(k)}_\beta$ and $M^{(k)}_\beta$ is sufficient
to compute the performance of $f^{(k)}$
(i.e., to compute $D^{(k)}_\beta$ and $N^{(k)}_\beta$).

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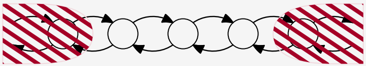
Step 2 Computing $L_\beta^{(k)}$ and $M_\beta^{(k)}$

Markov chain setup



$$L_\beta^{(k)}(e) = d(e) + \beta \sum_{n=-k}^{k} p_{n-e} L_\beta^{(k)}(n)$$

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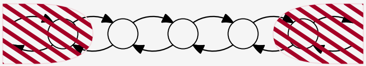

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Proposition

$$L_\beta^{(k)} = \Big[[I - \beta P^{(k)}]^{-1} d^{(k)}\Big].$$

$P^{(k)}$ is substochastic.

$$M_\beta^{(k)} = \Big[[I - \beta P^{(k)}]^{-1} \mathbf{1}^{(k)}\Big].$$

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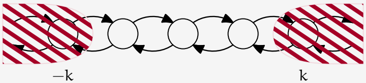

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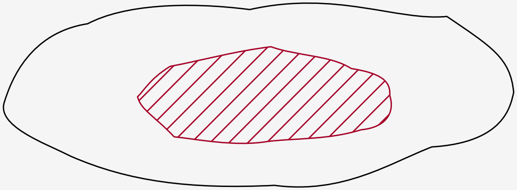
$D_\beta^{(k)}$ and $N_\beta^{(k)}$ can be computed using these expressions.

We found the performance of a generic threshold-based strategy

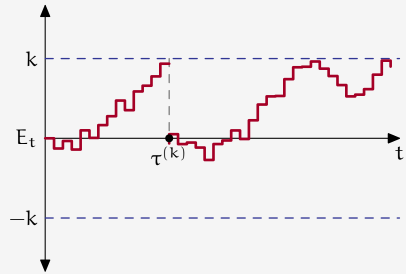
How does this lead to identifying an optimal strategy?

Step 1 Structure of optimal strategies

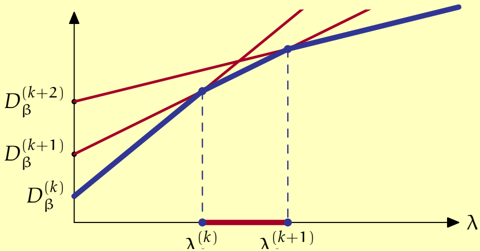


Search space of strategies (f, g)

Step 2 Performance of arbitrary threshold strategies $f^{(k)}$



Step 3 Optimal costly comm.



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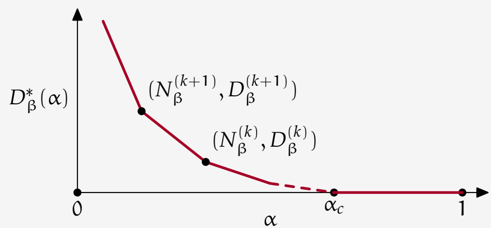

Step 3 Properties of optimal thresholds

Monotonicity $\quad L_{\beta}^{(k+1)} > L_{\beta}^{(k)}$ and $M_{\beta}^{(k+1)} > M_{\beta}^{(k)}$

Depends on unimodularity of noise

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Monotonicity $L_\beta^{(k+1)} > L_\beta^{(k)}$ and $M_\beta^{(k+1)} > M_\beta^{(k)}$

Use DP and monotonicity of Bellman operator

Implication:

$D_\beta^{(k+1)} \geqslant D_\beta^{(k)}$ and $N_\beta^{(k+1)} < N_\beta^{(k)}$

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$k_\beta^*(\lambda) \coloneqq \arg\min_{k \in \mathbb{Z}_{\geqslant 0}} C_\beta^{(k)}(\lambda)$ is increasing in λ.

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Proposition $k_\beta^*(\lambda) := \arg\min_{k \in \mathbb{Z}_{\geqslant 0}} C_\beta^{(k)}(\lambda)$ is increasing in λ.

Thus, optimal threshold increases with increase in λ.

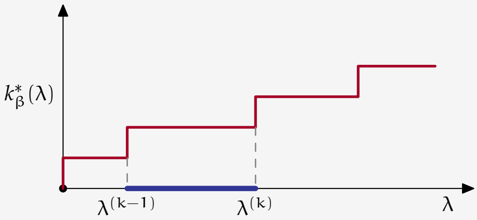
Characterizing the optimal threshold for a given communication cost **is tricky.**

Instead, we will characterize the optimal communication cost for a given threshold.

Step 3 Optimal costly communication: Markov chain

Define $\Lambda_\beta^{(k)} := \{\lambda \in \mathbb{R}_{\geqslant 0} : k_\beta^*(\lambda) = k\}$
$= [\lambda_\beta^{(k-1)}, \lambda_\beta^{(k)}]$.

$$C_\beta^{(k)}(\lambda_\beta^{(k)}) = C_\beta^{(k+1)}(\lambda_\beta^{(k)})$$



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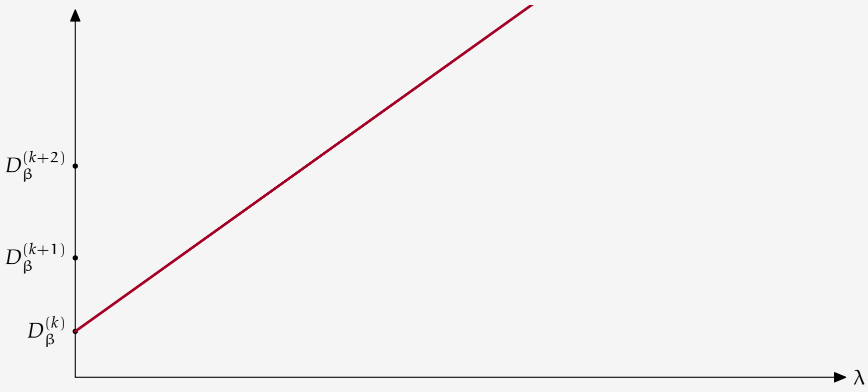

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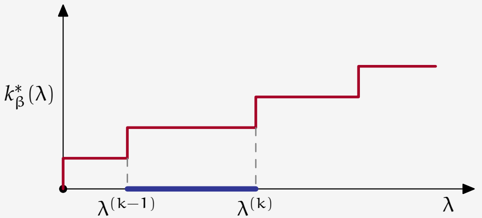

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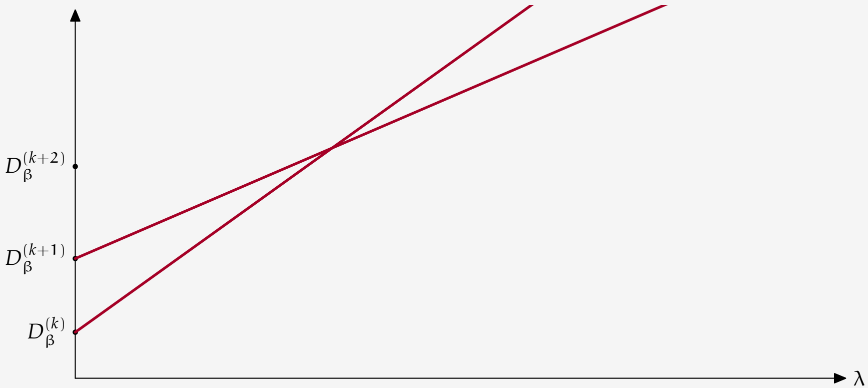

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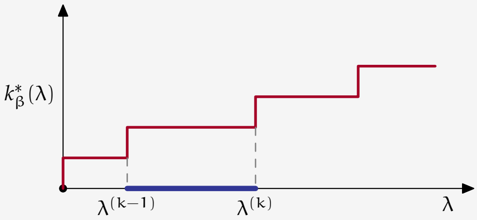

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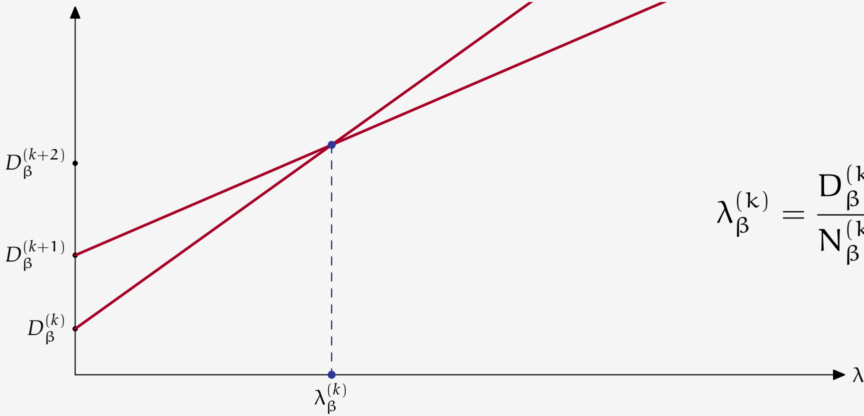

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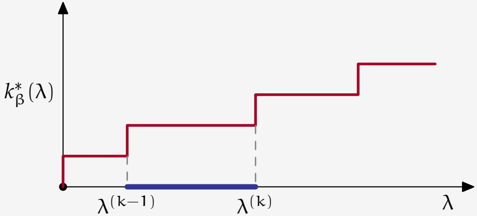

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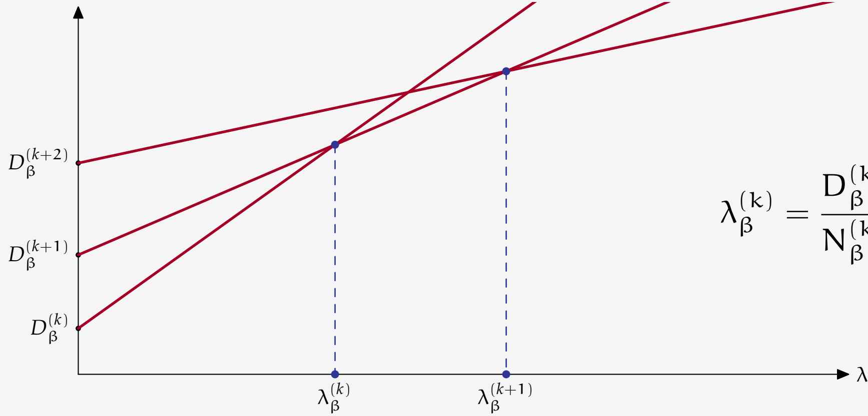

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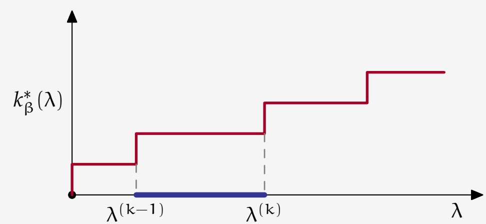

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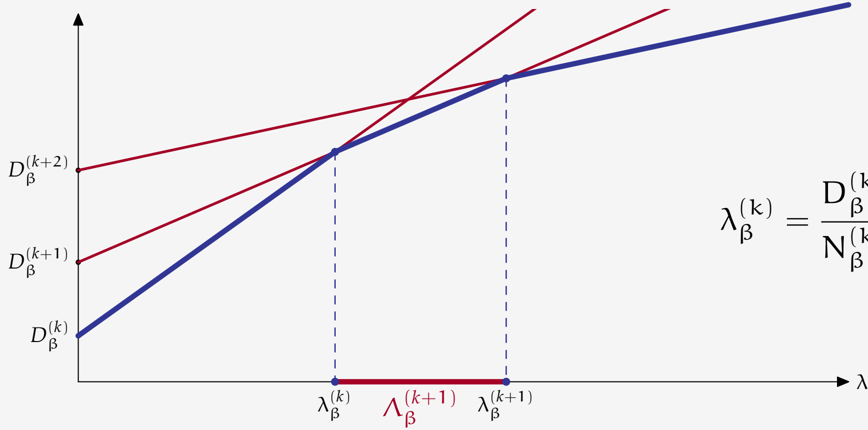

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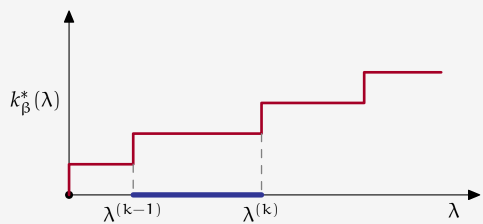

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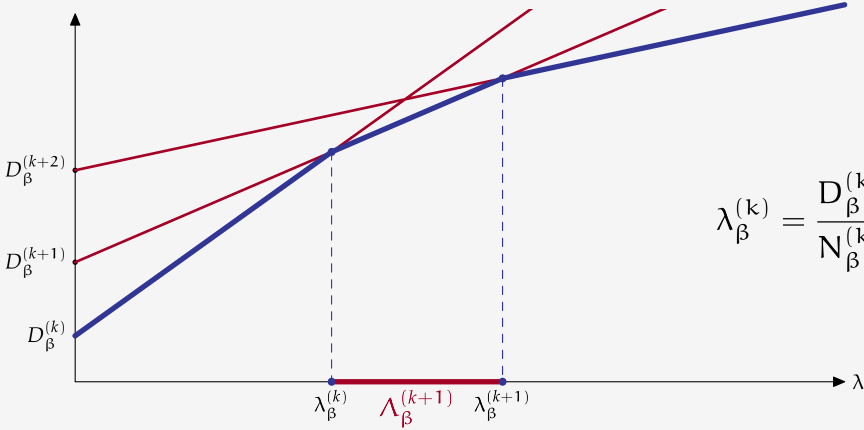

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Theorem Strategy $f^{(k+1)}$ is optimal for $\lambda \in (\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}]$.

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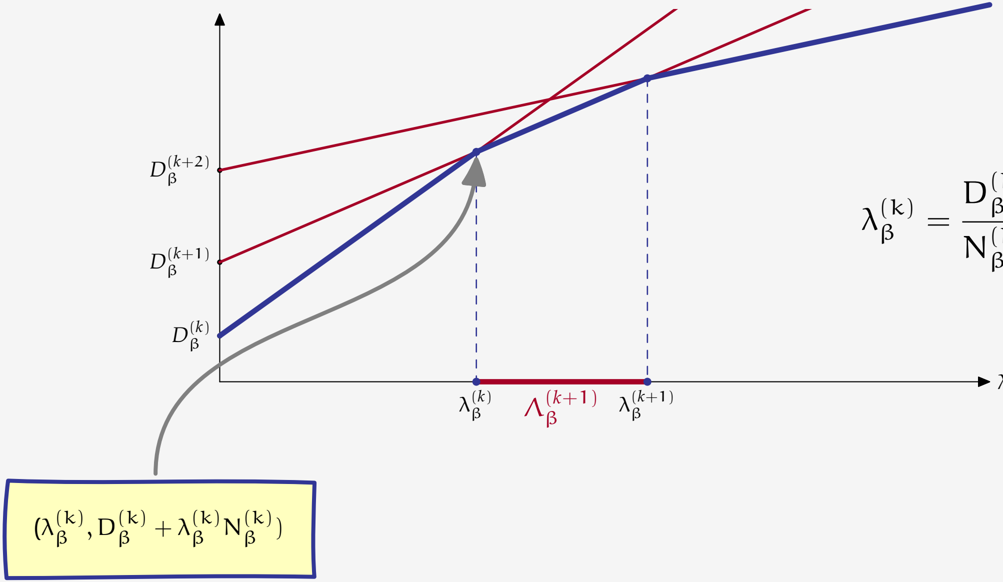

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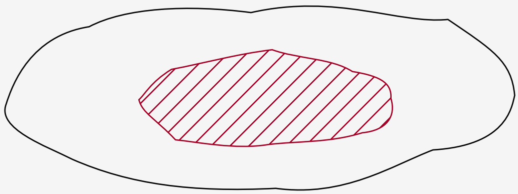

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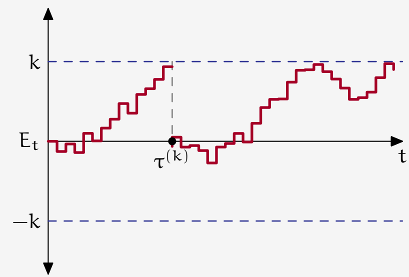

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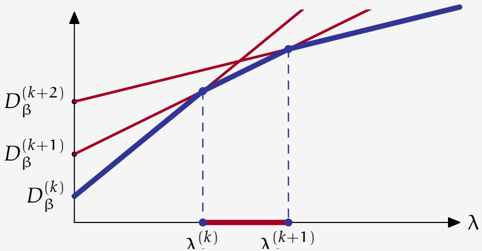

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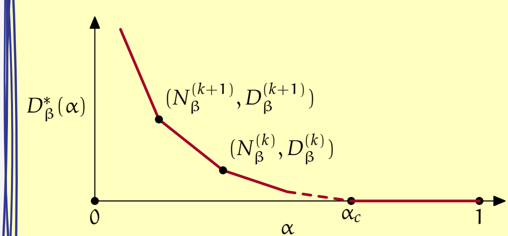

Step 4 Distortion-transmission trade-off: Markov chain

Sufficient conditions for constrained optimality

A strategy (f°, g°) is optimal for the constrained communication problem if

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$

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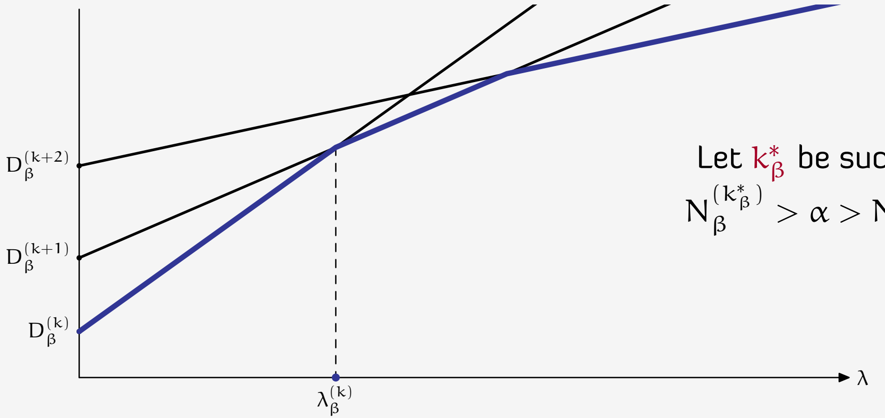

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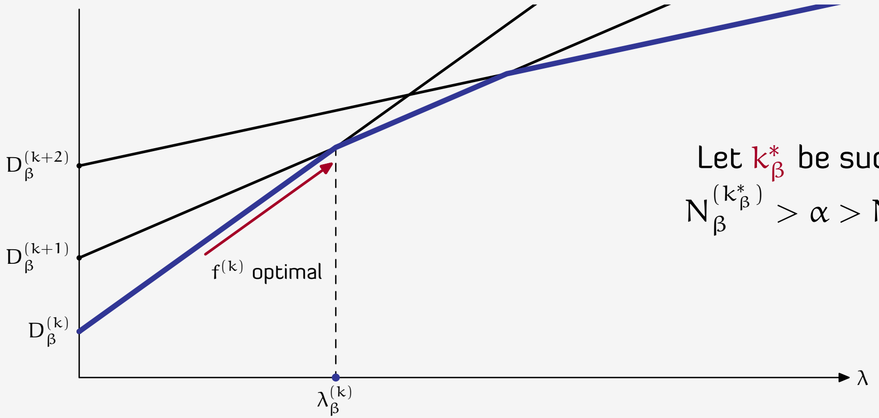

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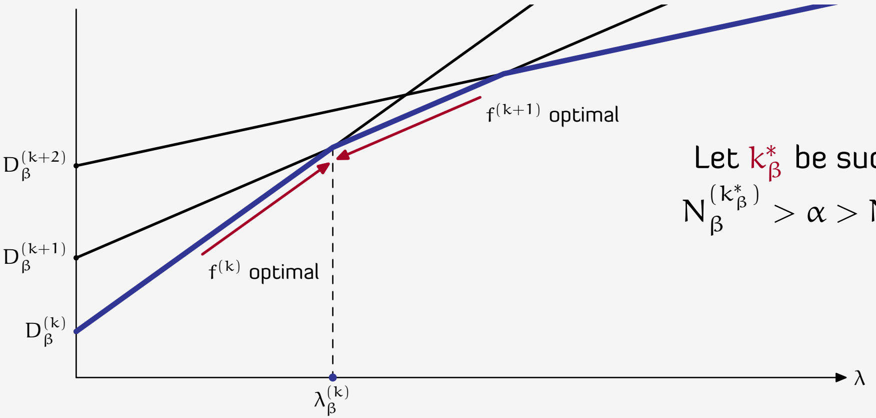

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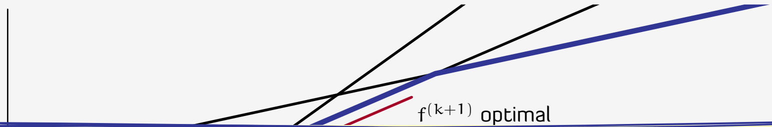

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Sufficient conditions for constrained optimality

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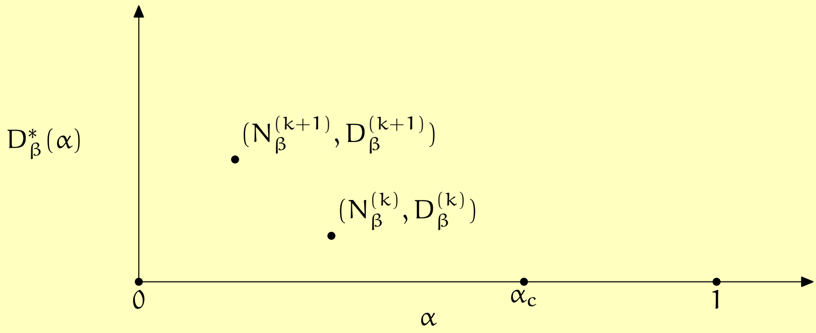

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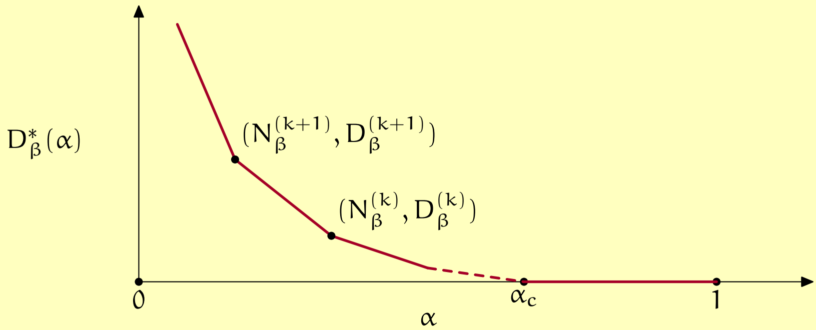

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A simple non-trivial "toy-problem" for decentralized control

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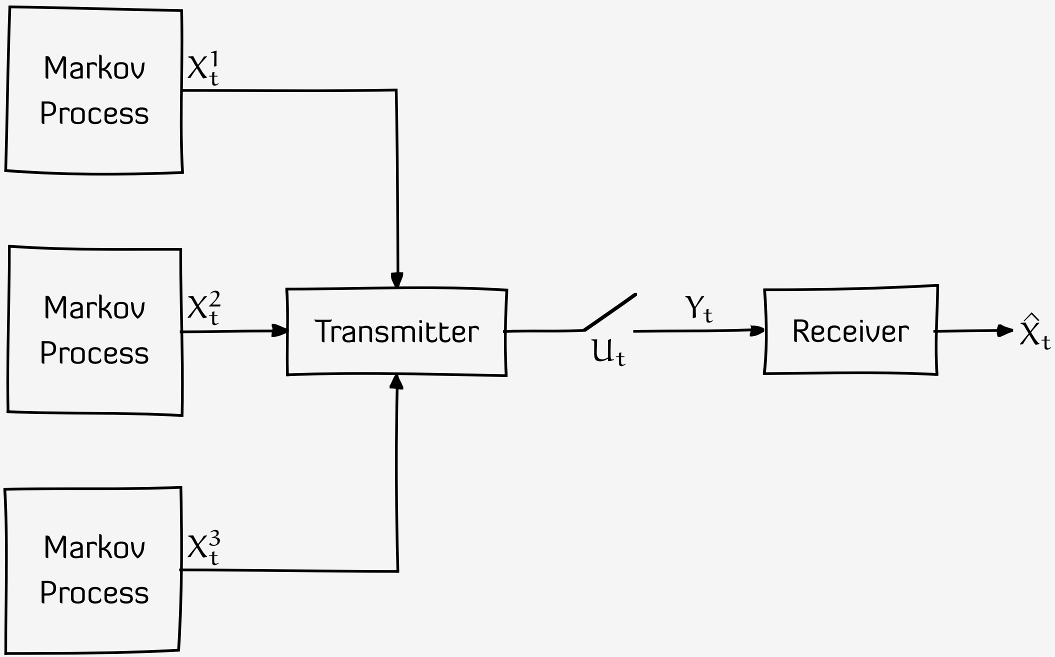
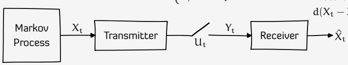
Full version available at arXiv:1505.04829.

A bandit variation



The system model

$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \varepsilon, & \text{if } U_t = 0 \end{cases}$$



$U_t = f_t(X_{1:t}, U_{1:t-1})$ $\qquad$ $\hat{X}_t = g_t(Y_{1:t})$

1. Discounted setup, $\beta \in (0,1)$

$$D_\beta(f,g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t)\Big]; \qquad N_\beta(f,g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t\Big]$$

2. Average cost setup, $\beta = 1$

$$D_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t)\Big]; \qquad N_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} U_t\Big]$$

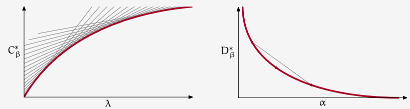
Optimization problems

Costly communication

For $\lambda \in \mathbb{R}_{>0}$, $\quad C^*_\beta(\lambda) = C_\beta(f^*, g^*; \lambda) := \inf_{(f,g)} \{D_\beta(f,g) + \lambda N_\beta(f,g)\}$

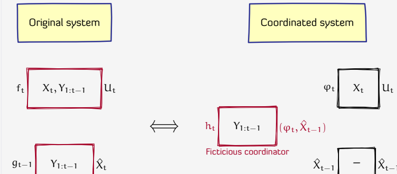
Constrained communication

For $\alpha \in (0,1)$, $\quad D^*_\beta(\alpha) := \inf_{(f,g)} \{D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha\}$



C^*_β is cts, inc, and concave $\qquad$ D^*_β is cts, dec, and convex

The common information approach

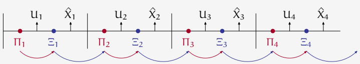


- The coordinated system is equivalent to the original system.
  $f_t(x, y_{1:t-1}) = h_t^1(y_{1:t-1})(x)$.
- The coordinated system is centralized. $\quad$ Belief state $\mathbb{P}(X_t \mid Y_{1:t-1})$.

Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Information states and dynamic program

Information states $\qquad$ Pre-transmission belief $\;: \Pi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t-1})$.
Post-transmission belief $: \Xi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t})$.



Structural results $\qquad$ There is no loss of optimality in using

$$U_t = f_t(X_t, \Pi_t) \quad \text{and} \quad \hat{X}_t = g_t(\Xi_t).$$

Dynamic Program $\qquad$ $W_{T+1}(\pi) = 0$

and for $t = T, \dots, 0$

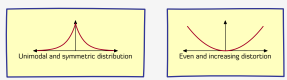
$$V_t(\xi) = \min_{\hat{x} \in \mathcal{X}} \mathbb{E}[d(X_t - \hat{x}) + W_{t+1}(\Pi_{t+1}) \mid \Xi_t = \xi],$$

$$W_t(\pi) = \min_{\varphi: \mathcal{X} \to \{0,1\}} \mathbb{E}[\lambda\varphi(X_t) + V_t(\Xi_t) \mid \Pi_t = \pi, \varphi_t = \varphi].$$

Simplifying modeling assumptions

Markov process $\qquad X_{t+1} = aX_t + W_t$

	Markov chain setup	**Guass-Markov setup**
State spaces	X_t, a, $W_t \in \mathbb{Z}$	X_t, a, $W_t \in \mathbb{R}$
Noise distribution	Unimodal and symmetric $p_e = p_{-e} \geqslant p_{e+1}$	Zero-mean Gaussian $\varphi_\sigma(\cdot)$
Distortion	Even and increasing $d(e) = d(-e) \leqslant d(e+1)$	Mean-squared $d(e) = \lvert e \rvert^2$



Step 1 Structure of optimal strategies (finite horizon)

Oblivious estimation process $\qquad Z_t = \begin{cases} X_t & \text{if } U_t = 1 \text{ (or } Y_t \neq \varepsilon) \\ aZ_{t-1} & \text{if } U_t = 0 \text{ (or } Y_t = \varepsilon) \end{cases}$

Error process $\qquad E_t = X_t - aZ_{t-1}$

Optimal estimator $\qquad \hat{X}_t = g_t^*(Z_t) = Z_t$

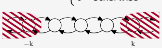
Optimal transmitter $\qquad$ There exists thresholds $\{k_t\}_{t=0}^{\infty}$ such that

$$U_t = f_t^*(E_t) = \begin{cases} 1 & \text{if } \lvert E_t \rvert \geqslant k_t \\ 0 & \text{if } \lvert E_t \rvert < k_t \end{cases}$$

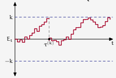
Step 2 Performance of threshold strategies

Consider a threshold-based strategy

$$f^{(k)}(e) = \begin{cases} 1 & \text{if } \lvert e \rvert \geqslant k \\ 0 & \text{otherwise} \end{cases}$$



Let $\tau^{(k)}$ denote the stopping time of first transmission (starting at $E_0 = 0$).



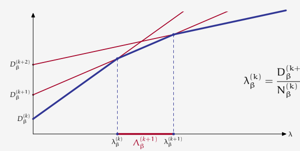
Define $\qquad L_\beta^{(k)}(e) = (1-\beta)\,\mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = e\Big].$

$$M_\beta^{(k)}(e) = (1-\beta)\,\mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \Big| E_0 = e\Big].$$

Proposition $\qquad \{E_t\}_{t=0}^{\infty}$ is a regenerative process. By renewal theory,

$$D_\beta^{(k)} := D_\beta(f^{(k)}, g^*) = \frac{L_\beta^{(k)}(0)}{M_\beta^{(k)}(0)} \quad \text{and} \quad N_\beta^{(k)} := N_\beta(f^{(k)}, g^*) = \frac{1}{M_\beta^{(k)}(0)} - (1-\beta).$$

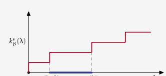
Step 3 Optimal costly communication: Markov chain



$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

Define $\Lambda_\beta^{(k)} := \{\lambda \in \mathbb{R}_{\geqslant 0} : k_\beta^*(\lambda) = k\}$
$= [\lambda_\beta^{(k-1)}, \lambda_\beta^{(k)}]$.

$$C_\beta^{(k)}(\lambda_\beta^{(k)}) = C_\beta^{(k+1)}(\lambda_\beta^{(k)})$$



Step 4 Distortion-transmission trade-off: Markov chain

Sufficient conditions for constrained optimality

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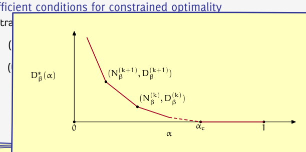

Randomized strategy $(\theta^*, f^{(k)}, f^{k+1})$ is optimal where

$$\theta^* N_\beta^{(k)} + (1-\theta^*) N_\beta^{(k+1)} = \alpha$$