Consistency and Rate of Convergence of Switched Least Squares System Identification for Autonomous Markov Jump Linear Systems

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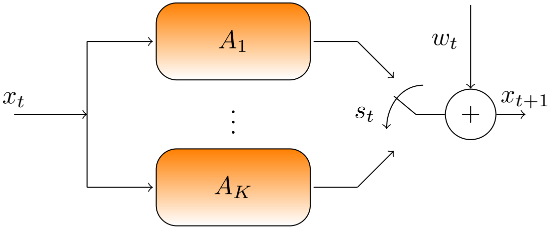
Motivation



- Markov jump linear systems are suitable in modeling dynamical systems with abrupt changes.

Application of Markov jump linear systems

- Network Control Systems (NCS)
- Power systems
- Cyber-physical systems
- Time series analysis (e.g. in stock markets)

- **Question**: How we can perform system identification?

System Identification for Linear Systems

- We use a least squares based algorithm

Convergence of Least squares method (a.s. results)

1. Convergence of general regression: [Lai and Wei, 1982]
2. Autonomous linear systems: [Lai and Wei, 1985]
3. Least squares for ARMAX models : [Chen and Guo, 1986, Chen and Guo, 1987, Lai and Wei, 1986, Caines, 2018]

Convergence of Least squares method (high probability bounds)

1. [Abbasi-Yadkori and Szepesvári, 2011, Faradonbeh et al., 2018, Simchowitz et al., 2018, Oymak and Ozay, 2019, Lale et al., 2020]

Literature Review

- Adaptive control for switched linear systems:

Asymptotically stable adaptive controllers

1. [Caines and Chen, 1985, Caines and Zhang, 1995, Xue and Guo, 2001]

- The system identification of Markov/switched least squares problem is less explored.

System identification for switched linear systems

1. Switched linear systems: [Ozay et al., 2015, Hespanhol and Aswani, 2020, Sarkar et al., 2019, Shi et al., 2023]
2. Markov jump linear systems: [Sattar et al., 2021]

Contributions

Our results

- *Switched least squares* for system Identification of MJS.
- Converges *almost surely* and establish *strong consistency*.
- Data-dependent and data-independent rate of convergence.

Compared to the literature

1. Convergence guarantees:
   - Almost sure convergence/high probability convergence
2. Assumption on stability:
   - Stability on the average sense/Mean square stability

An overview of the results

1. Theorem 1: (Strong consistency)

$$\lim_{T\to\infty}\left\|\hat{A}_{i,T}-A_i\right\|_\infty=0,\quad \text{a.s.}\quad \forall i$$

2. Theorem 2: (Rate of convergence)
   - Data-dependent.
   - Data-independent.

$$\left\|\hat{A}_{i,T}-A_i\right\|_\infty\leq\mathcal{O}\left(\sqrt{\log(T)/T}\right),\quad \text{a.s.}\quad \forall i$$

Autonomous Markov jump linear systems

- states : (s_t, x_t), $s_t \in \{1, \ldots, k\}$, $x_t \in \mathbb{R}^n$
- set of possible modes : $\mathcal{A} = \{A_1, \ldots, A_k\}$.

Continuous state dynamics

$$x_{t+1} = A_{s_t} x_t + w_t, \quad t \geq 0,$$

where $\{w_t\}_{t\geq 0}$, $w_t \in \mathbb{R}^n$, is a noise process.

Discrete state dynamics

- $\{s_t\}_{t\geq 0}$ is a time-homogeneous irreducible and aperiodic Markov chain.
- $\pi_\infty = (\pi_\infty(1), \ldots, \pi_\infty(k))$: stationary distribution $(\pi(i) \neq 0)$.

Assumption on the noise process

- Let $\mathcal{F}_{t-1} = \sigma(x_{0:t}, s_{0:t})$.

Assumption on the noise

1. $\{w_t\}_{t \geq 0}$ is a martingale difference sequence w.r.t. $\{\mathcal{F}_t\}_{\geq 0}$ i.e.:

$$\mathbb{E}[|w_t|] < \infty \quad \text{and} \quad \mathbb{E}[w_t \mid \mathcal{F}_{t-1}] = 0$$

2. Furthermore, $\exists\ \alpha > 2$ such that : $\sup_{t \geq 0} \mathbb{E}[\|w_t\|^{\alpha} \mid \mathcal{F}_{t-1}] < \infty \quad$ a.s.
3. There exists $0 \prec C \in \mathbb{R}^{n \times n}$ such that:

$$\liminf_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} w_t w_t^{\intercal} = C$$

- Noise process can be non-stationary and heavy tailed process.

Assumption on the stability

Assumption on the stability of system

The MJS system is stable in the *average sense* if:

$$\sum_{t=1}^{T} \|x_t\|^2 = \mathcal{O}(T) \quad \text{a.s.}$$

Assumption: The MJS system is stable in the average sense.

Switched Least Squares Method

- Global Least squares optimization:

$$\hat{\theta}_T^{\intercal} = \underset{\theta^{\intercal}=[A_1,\ldots,A_k]}{\arg\min} \sum_{t=0}^{T-1} \|x_{t+1} - A_{s_t} x_t\|^2.$$

- Suppose: $\mathcal{T}_{i,T} = \{t \leq T | s_t = i\}$.

Switched Least Squares Problem

- Decoupling:

$$\hat{A}_{i,T} = \underset{A_i \in \mathbb{R}^{n\times n}}{\arg\min} \sum_{t\in\mathcal{T}_{i,T}} \|x_{t+1} - A_i x_t\|^2, \quad \forall i \in \{1, \cdots, k\}.$$

A few definitions

- $X_{i,T} = \sum_{t \in \mathcal{T}_{i,T}} x_t x_t^\intercal$: i-th mode empirical covariance
- The least squares update:

$$\hat{A}_{i,T+1} = \hat{A}_{i,T} + \left[\frac{X_{i,T}^{-1} x_T (x_{T+1} - \hat{A}_{i,T} x_T)^\intercal}{1 + x_T^\intercal X_{i,T}^{-1} x_T} \right] \mathbb{1}\{s_{T+1} = i\}$$

where: $X_{i,T+1} = X_{i,T} + \left[x_{T+1} x_{T+1}^\intercal \right] \mathbb{1}\{s_{T+1} = i\}$.

Strong consistency

- An estimator $\hat{\theta}_T$ is called strongly consistent if:

$$\lim_{T \to \infty} \hat{\theta}_T = \theta \quad \text{a.s.}$$

Main Results (Data dependent)

Theorem

- *The switched least squares estimates $\{\hat{A}_{i,T}\}_{i=1}^{k}$ are strongly consistent, i.e.,:*

$$\lim_{T\to\infty} \left\| \hat{A}_{i,T} - A_i \right\|_\infty = 0, \quad a.s. \quad \forall i$$

- Rate of convergence:

$$\left\| \hat{A}_{i,T} - A_i \right\|_\infty \leq \mathcal{O}\left(\sqrt{\frac{\log\left[\lambda_{\max}(X_{i,T})\right]}{\lambda_{\min}(X_{i,T})}} \right), \quad \text{a.s.}$$

Main Results (Data independent)

Theorem

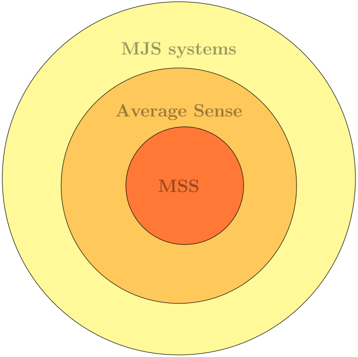
1. *The rate of convergence of the switched least squares is upper bounded by:*

$$\left\|\hat{A}_{i,T} - A_i\right\|_\infty \leq \mathcal{O}\left(\sqrt{\log(T)/\pi_\infty(i)T}\right), \quad a.s.$$

Mean square stability

- System is Mean square stable (MSS) if $\exists\, x_\infty, Q_\infty$:

$$\lim_{\tau\to\infty} \left\| \mathbb{E}[x_\tau] - x_\infty \right\| \to 0 \quad \text{and} \quad \lim_{\tau\to\infty} \left\| \mathbb{E}[x_\tau x_\tau^\intercal] - Q_\infty \right\| \to 0.$$



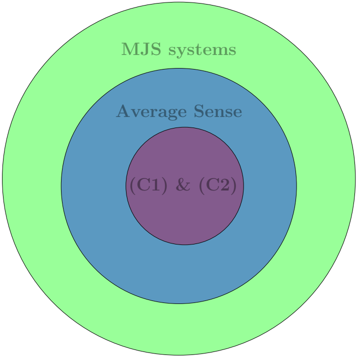
- $\sum_{t=1}^{T} \|x_t\|^2 = \mathcal{O}(T)$ a.s.

Results

(i) Average sense stability $\prec$ MSS
(ii) MSS $\implies$ Consistency

Almost sure stability

- Suppose π_∞ satisfies **(C1)** $\pi_\infty(i) \neq 0 \quad \forall i$, **(C2)** $\prod_{i=1}^{k} \sigma_i^{\pi_\infty(i)} < 1$.

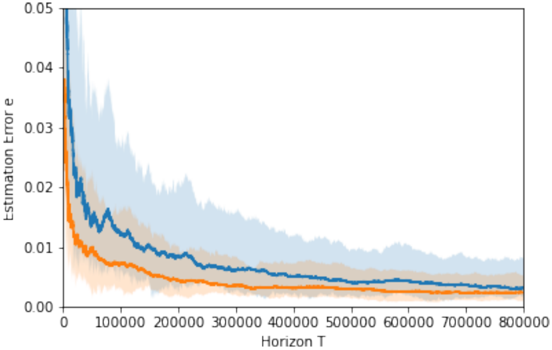


- $\sum_{t=1}^{T} \|x_t\|^2 = \mathcal{O}(T) \quad$ a.s.

Results

(i) Average sense stability $\prec$ (C1) & (C2)
(ii) (C1) & (C2) $\implies$ Consistency

Numerical simulation



- $A_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.2 \end{bmatrix}$, and $A_2 = \begin{bmatrix} 0.01 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$,
- transition matrix $P = \begin{bmatrix} 1/12 & 11/12 \\ 3/4 & 1/4 \end{bmatrix}$ with $\pi_\infty = [0.45, 0.55]$
- *Not* MSS and (C1) and (C2)

Conclusion

- *Strong* consistency of switched least squares for MJS systems.
- Data-independent rate of convergence of $\sqrt{\log(T)/T}$ almost surely.
- Stability in the average sense $\implies$ almost sure convergence of the method.
- Mean square stability $\implies$ almost sure convergence of the method .

Thank you!

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