

Renewal Monte Carlo:

Renewal theory based reinforcement learning

Jayakumar Subramanian and Aditya Mahajan

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December 17-19, 2018

RL has achieved considerable success...



Image credit: MIT Technology review



Image credit: Towards Data Science



Image credit: Popular Science

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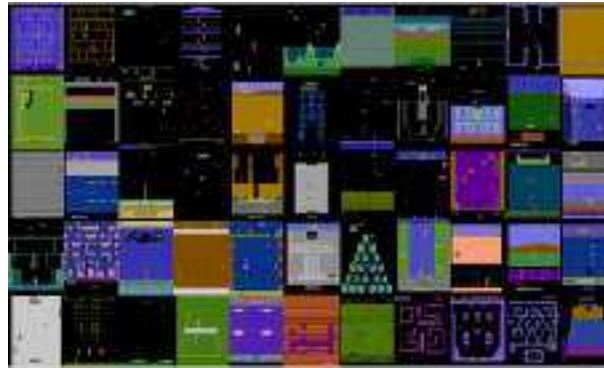


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Salient features

- ⊕ Model-free method
- ⊕ Use policy search

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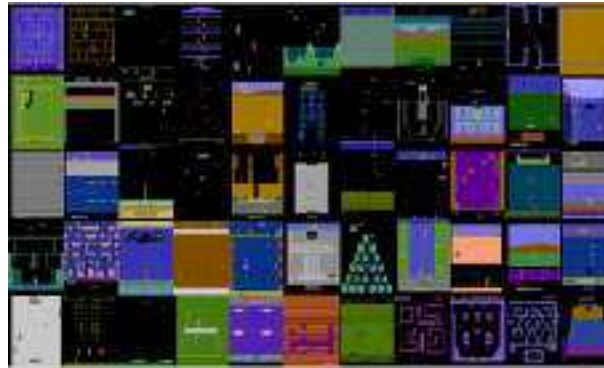


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Limitation

- ⊖ Learning is slow (takes $\sim 10^9$ to 10^{15} iterations to converge)

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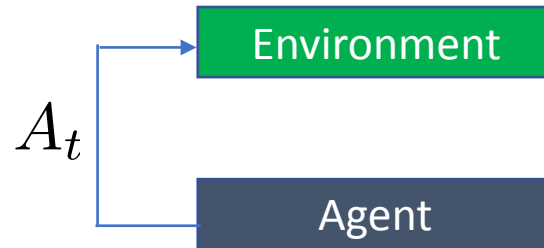
- ⊕ Can we exploit features of the model to make it learn faster? ...
- ⊕ Without sacrificing generality?

An RL problem can be formulated as...

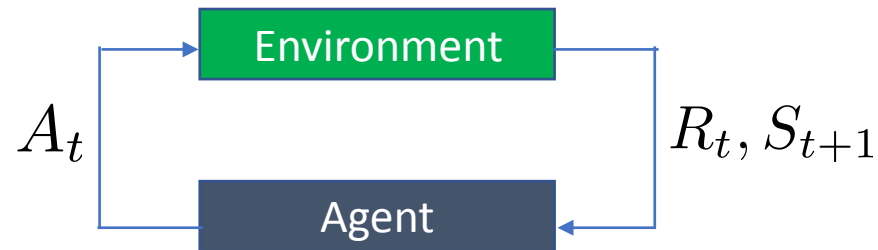
An RL problem can be formulated as...

Agent

An RL problem can be formulated as...



An RL problem can be formulated as...



An RL problem can be formulated as...



Infinite horizon Markov decision process (MDP) Model

State space

$$S_t \in \mathcal{S}$$

Action space

$$A_t \in \mathcal{A}$$

Transition probability

$$\mathbb{P}(S_{t+1}|S_t, A_t) = [P(A_t)]_{S_t, S_{t+1}}$$

Per-step reward

$$R_t = r(S_t, A_t, S_{t+1})$$

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Unknown in RL

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Policy parametrization

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Gibbs (softmax) policy

$$\mu_\theta(a|s) = \frac{\exp(\tau\theta(s, a))}{\sum_{a'} \exp(\tau\theta(s, a))}$$

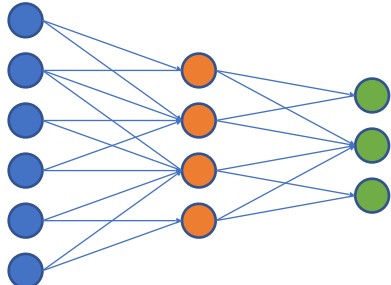
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Neural network (NN) policy

$$\mu_\theta(a|s) =$$


θ : weights of NN

Policy gradient

Policy gradient

Performance
Gradient
Estimate



Policy gradient

Performance
Gradient
Estimate

$$J_{\theta} = \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t R_t \mid S_0 = s_0, A_t \sim \mu_{\theta}(S_t) \right]$$

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How do we estimate this?

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Gradient
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Actor Critic estimate (Temporal difference / SARSA)

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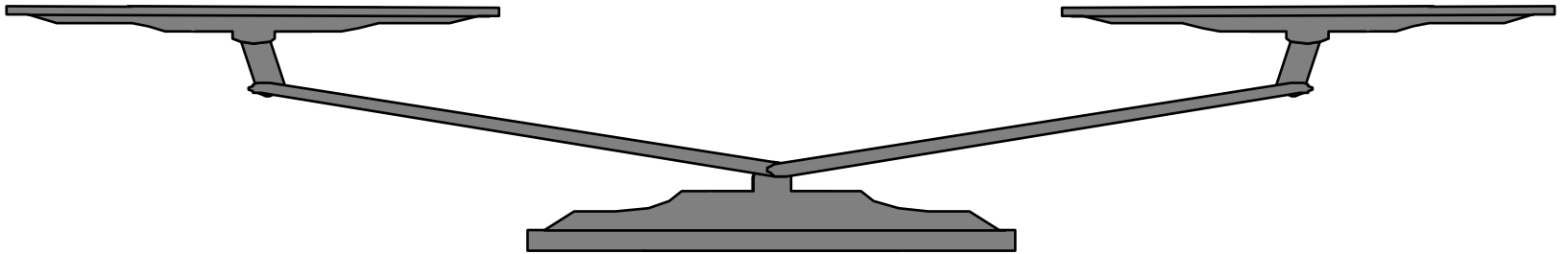
Actor Critic with eligibility traces estimate (SARSA- λ)

$$G_{\theta} = \sum_{t=0}^{\infty} \left[\nabla_{\theta} \log(\mu_{\theta}(A_t|S_t)) \gamma^t Q^{\lambda}(S_t, A_t) \right]$$

MC vs. TD

MC

TD

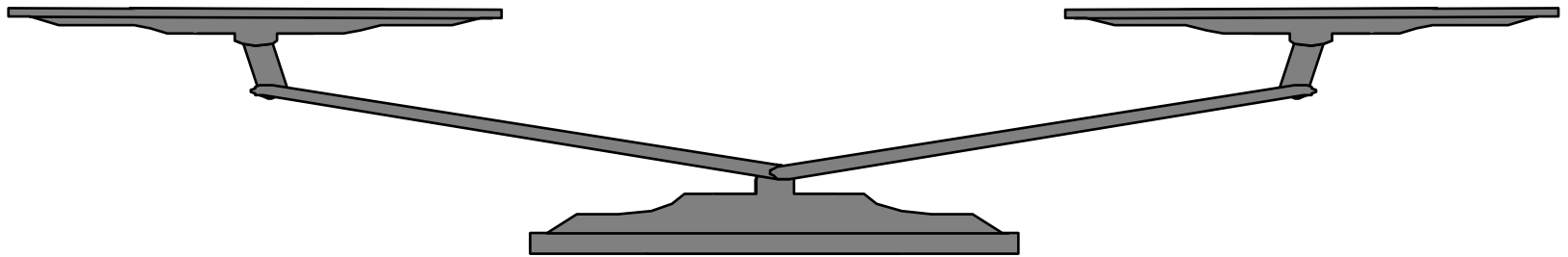


MC vs. TD

MC

- ⊕ Unbiased
- ⊕ Simple & easy to implement
- ⊕ Discounted & average reward cases
- ⊖ High variance
- ⊖ End-of-episode updates
- ⊖ Not asymptotically optimal for inf. hor.

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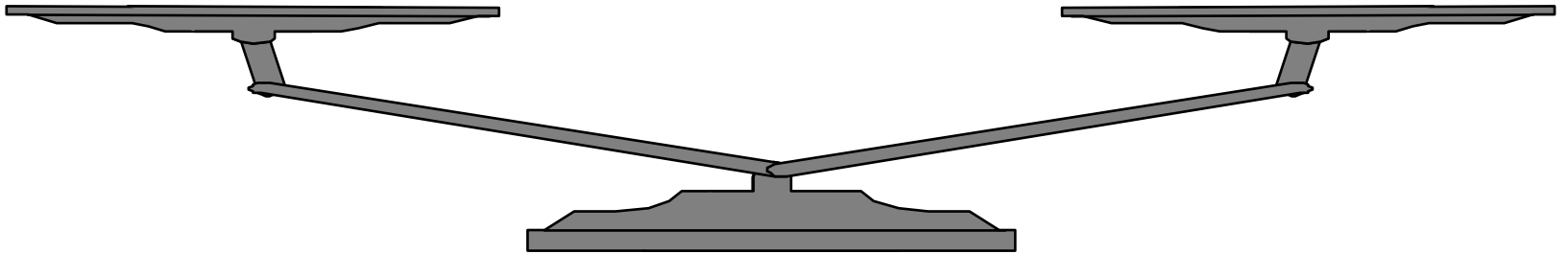
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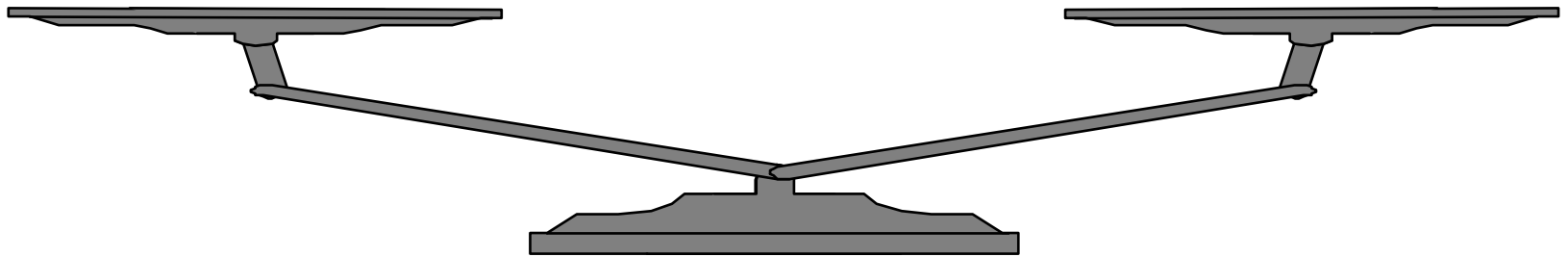
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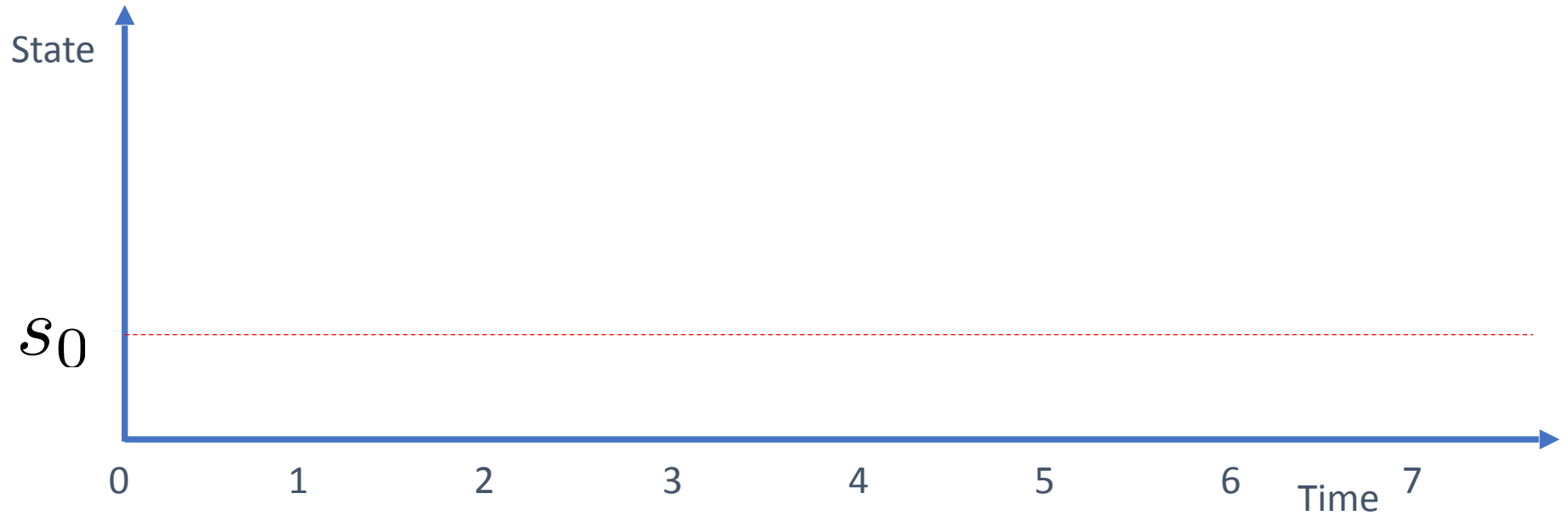
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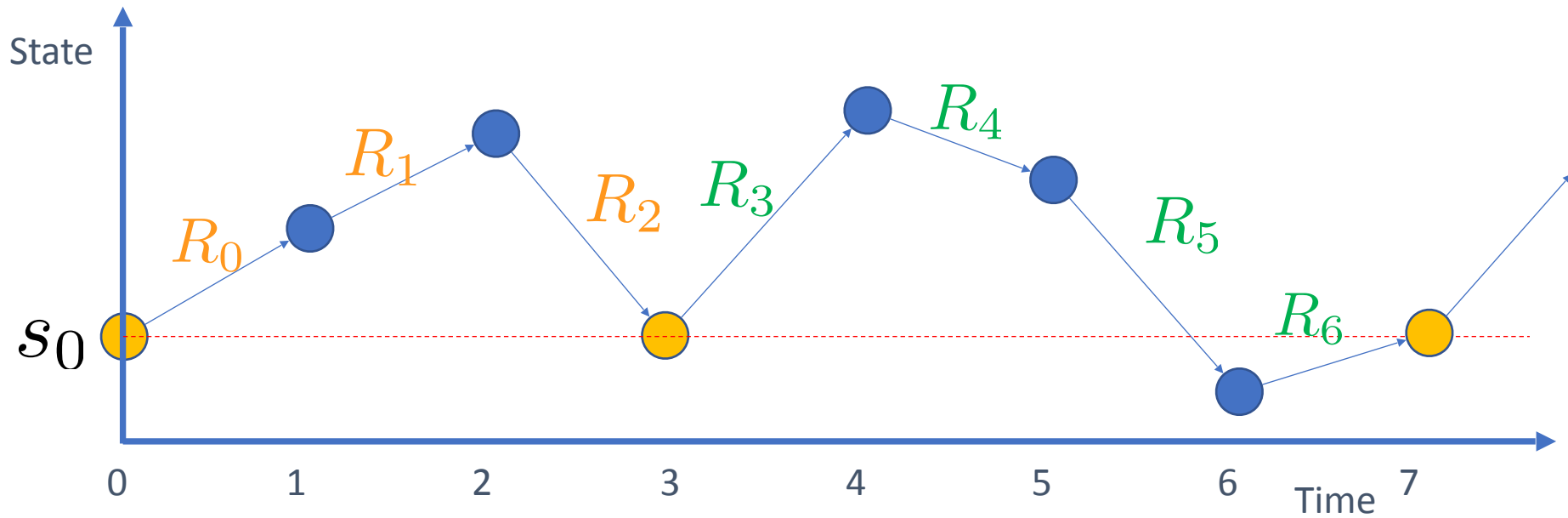


Can we get the best of both worlds?

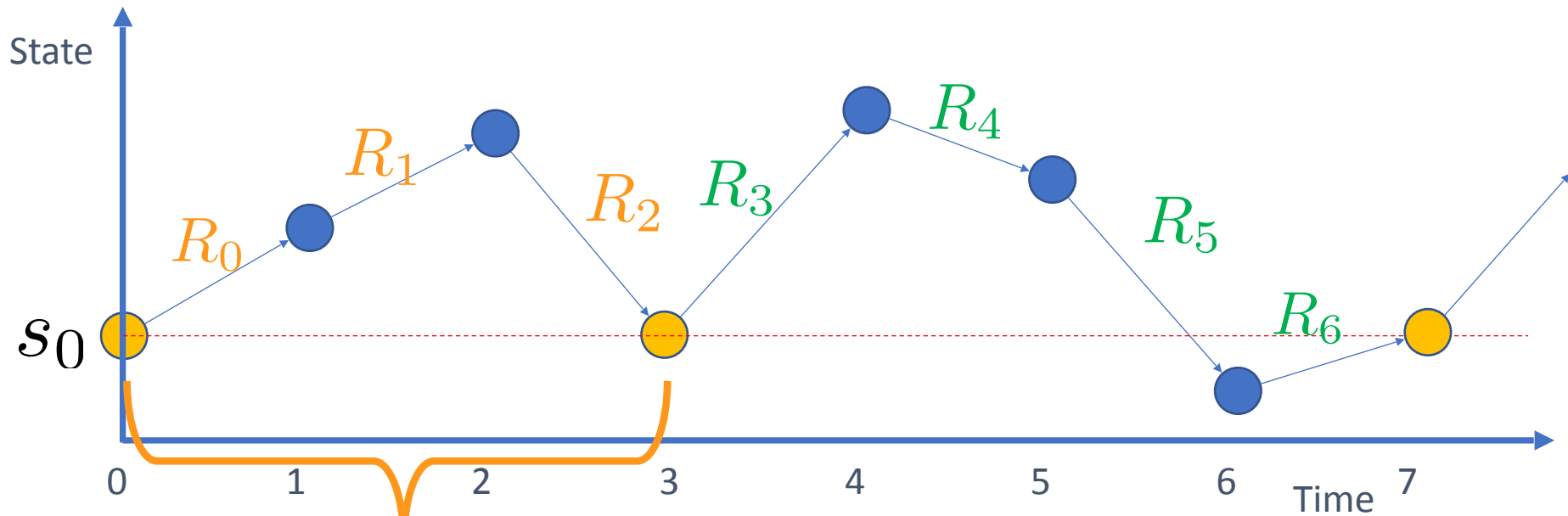
Renewal Monte Carlo



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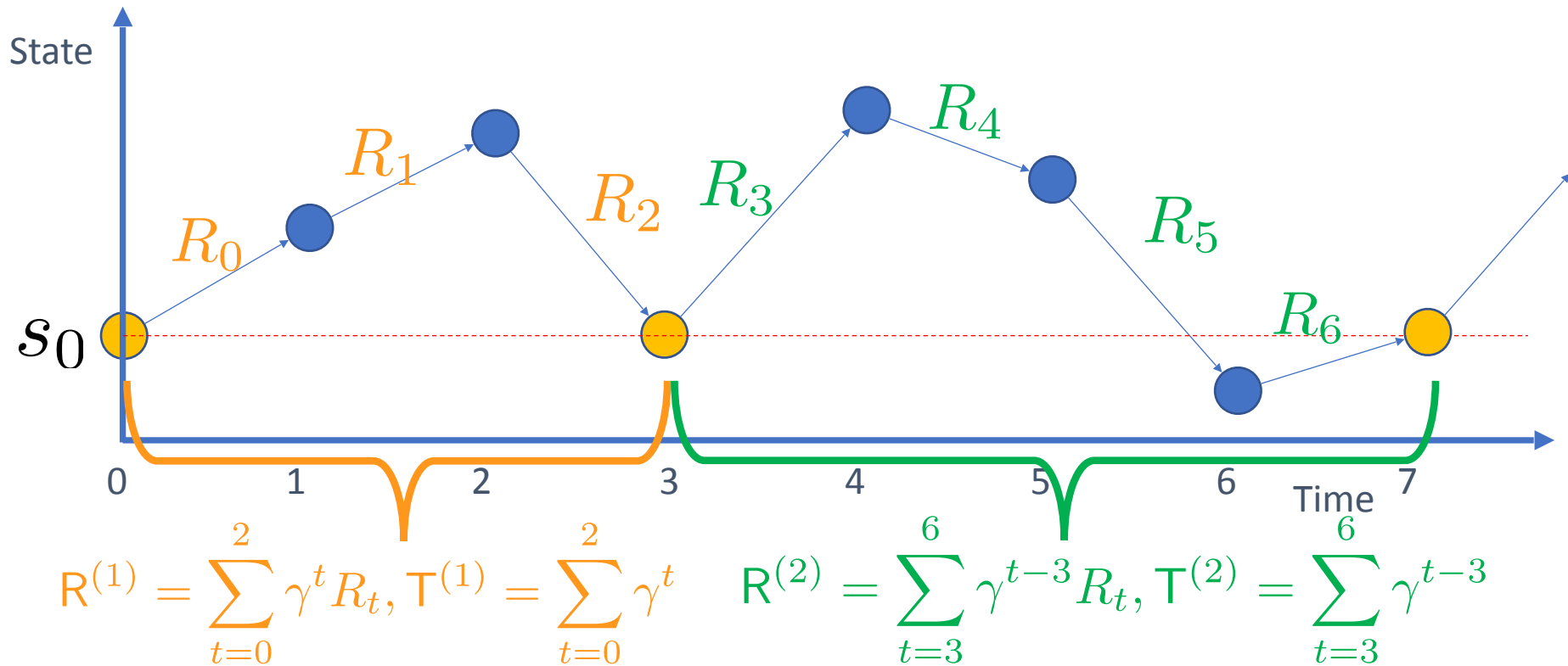


Renewal Monte Carlo

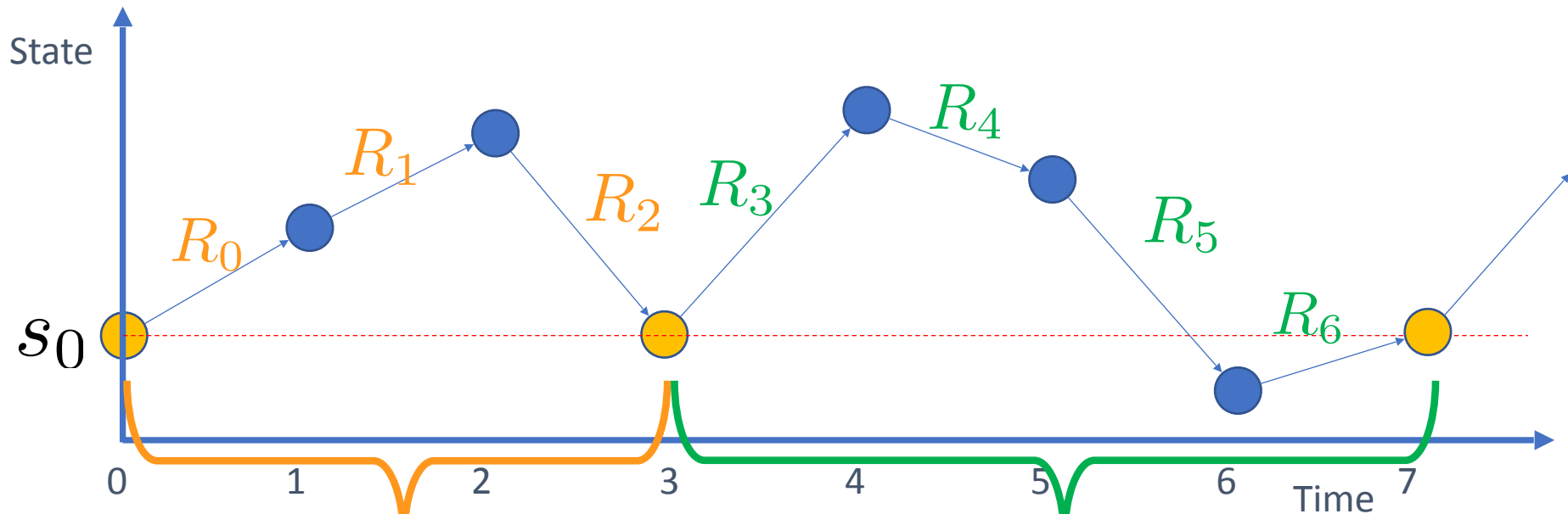


$$R^{(1)} = \sum_{t=0}^2 \gamma^t R_t, T^{(1)} = \sum_{t=0}^2 \gamma^t$$

Renewal Monte Carlo



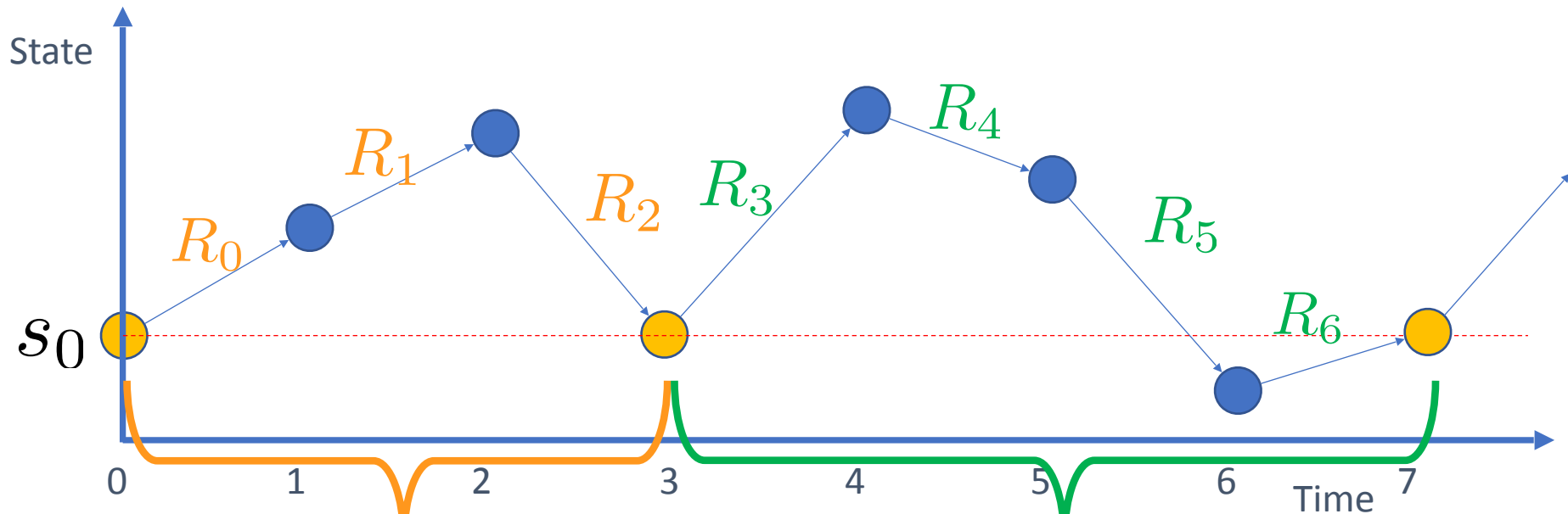
Renewal Monte Carlo



$$R^{(1)} = \sum_{t=0}^2 \gamma^t R_t, \quad T^{(1)} = \sum_{t=0}^2 \gamma^t \quad R^{(2)} = \sum_{t=3}^6 \gamma^{t-3} R_t, \quad T^{(2)} = \sum_{t=3}^6 \gamma^{t-3}$$

$R^{(1)}, R^{(2)} \dots$ are i.i.d and $T^{(1)}, T^{(2)} \dots$ are i.i.d

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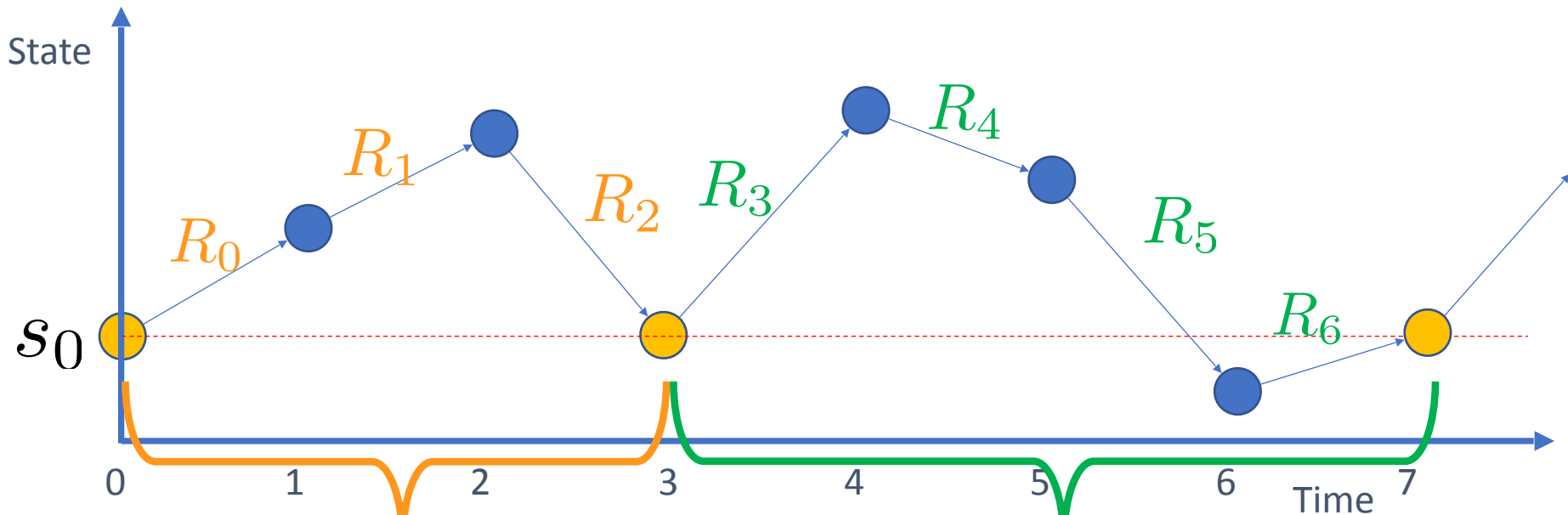


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$$R_\theta = \mathbb{E}[R^{(n)}] \text{ and } T_\theta = \mathbb{E}[T^{(n)}]$$

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$R_\theta = \mathbb{E}[R^{(n)}]$ and $T_\theta = \mathbb{E}[T^{(n)}]$ ▶ estimated by \hat{R}, \hat{T}

RMC based policy gradient

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Performance
Gradient
Estimate

$$J_{\theta} = \frac{R_{\theta}}{(1 - \gamma)T_{\theta}}$$

RMC based policy gradient

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$$J_{\theta} = \frac{R_{\theta}}{(1 - \gamma)T_{\theta}} \quad ; \quad \nabla_{\theta} J_{\theta} = \frac{H_{\theta}}{(1 - \gamma)T_{\theta}^2}$$

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$\hat{R}_{\theta}, \hat{T}_{\theta}$ estimated using MC / TD ; $\nabla_{\theta} \hat{R}_{\theta}, \nabla_{\theta} \hat{T}_{\theta}$ using RL policy gradient

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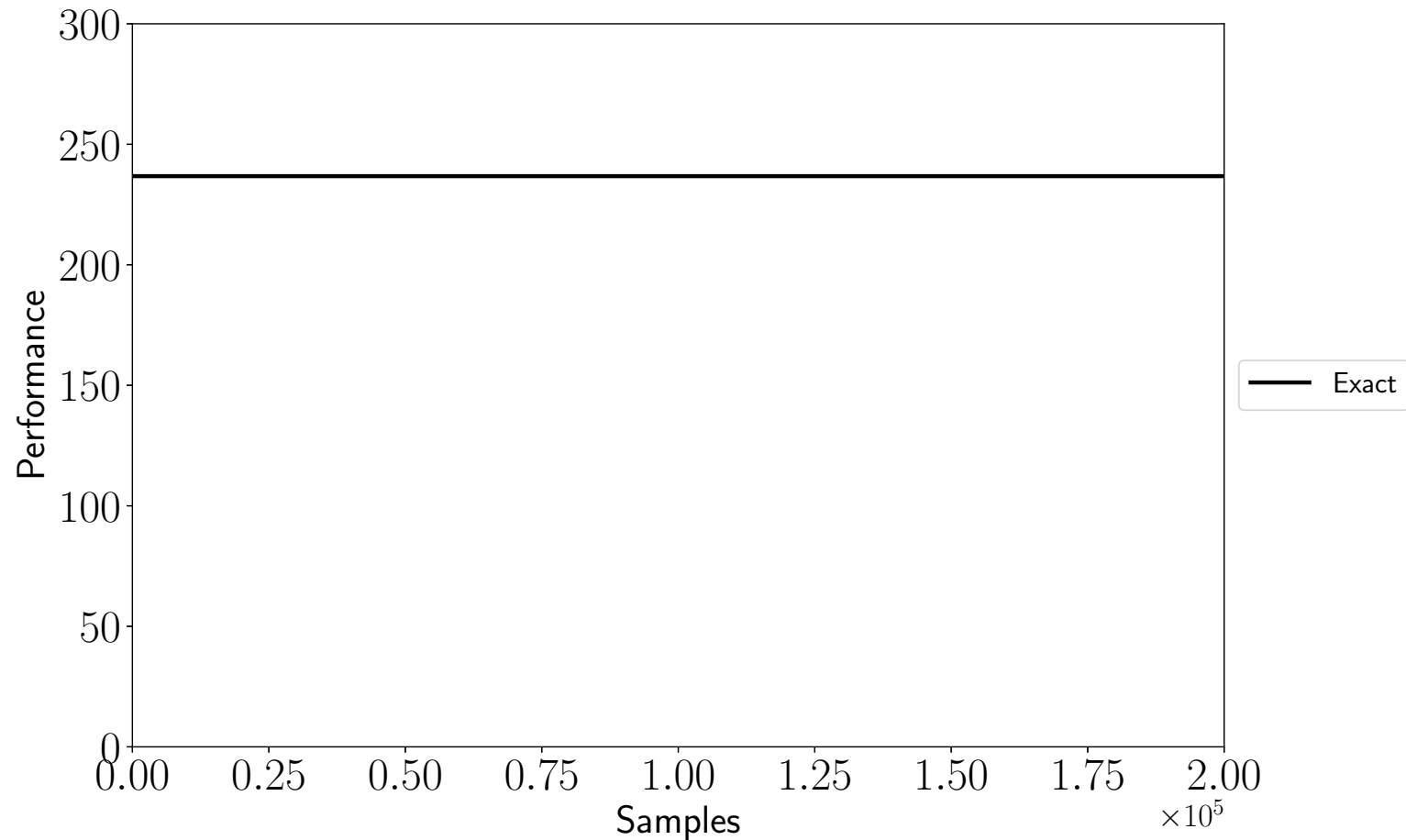
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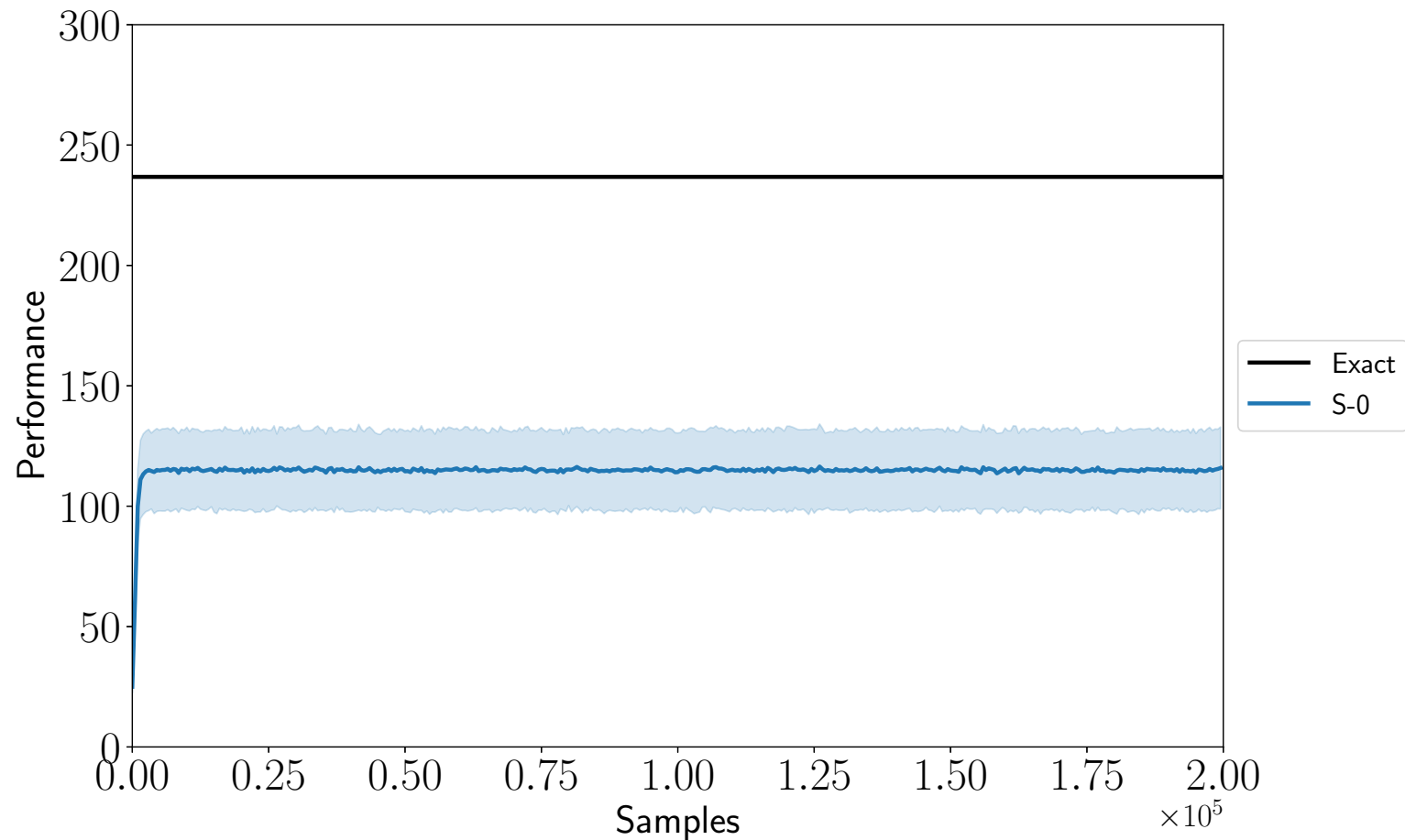
$\dot{\theta} = H_{\theta}$ has locally asymptotically stable isolated limit points

Iteration for θ_k converges a.s. to a value where $\nabla_{\theta} J_{\theta} = 0$

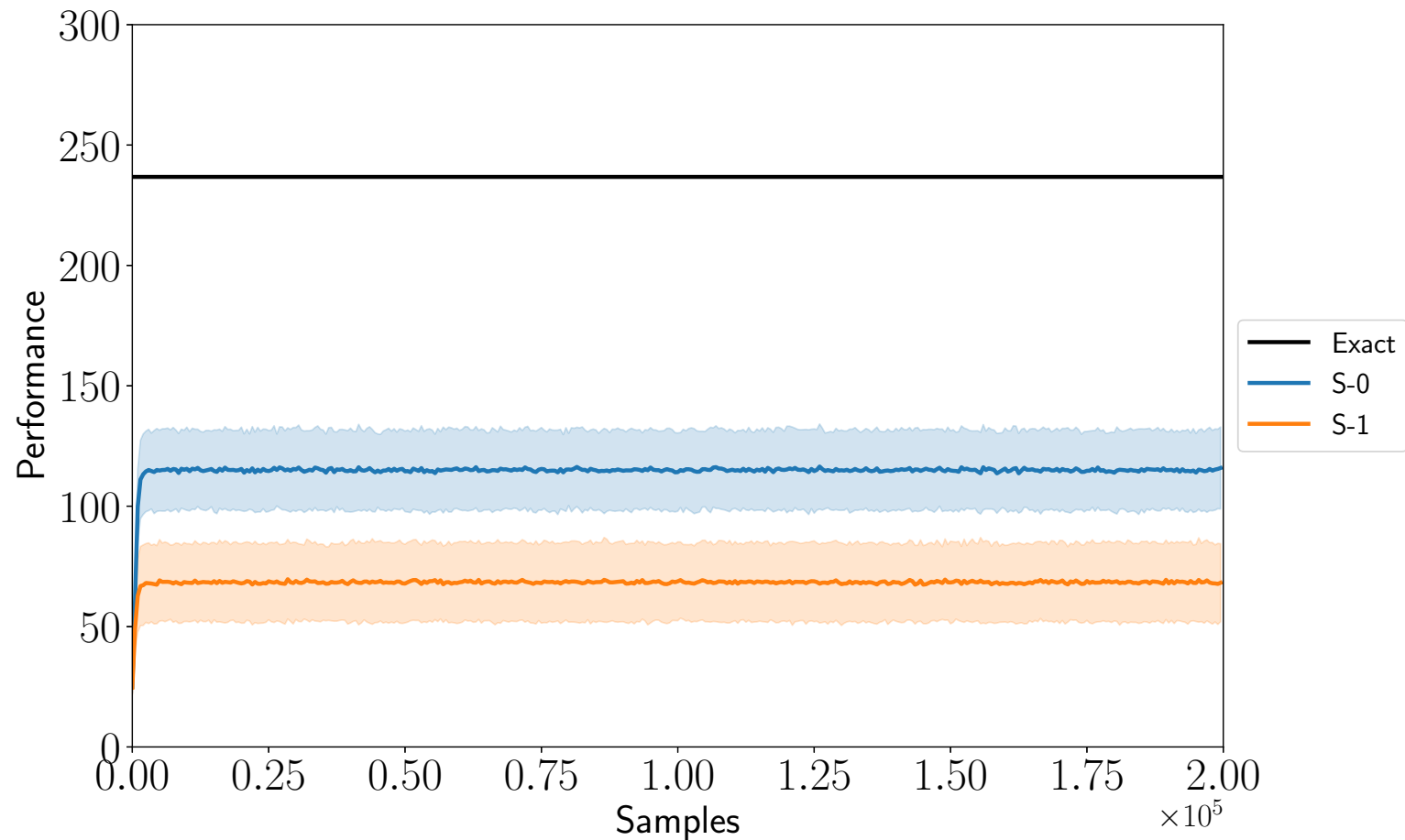
E.g. – Randomly generated MDP



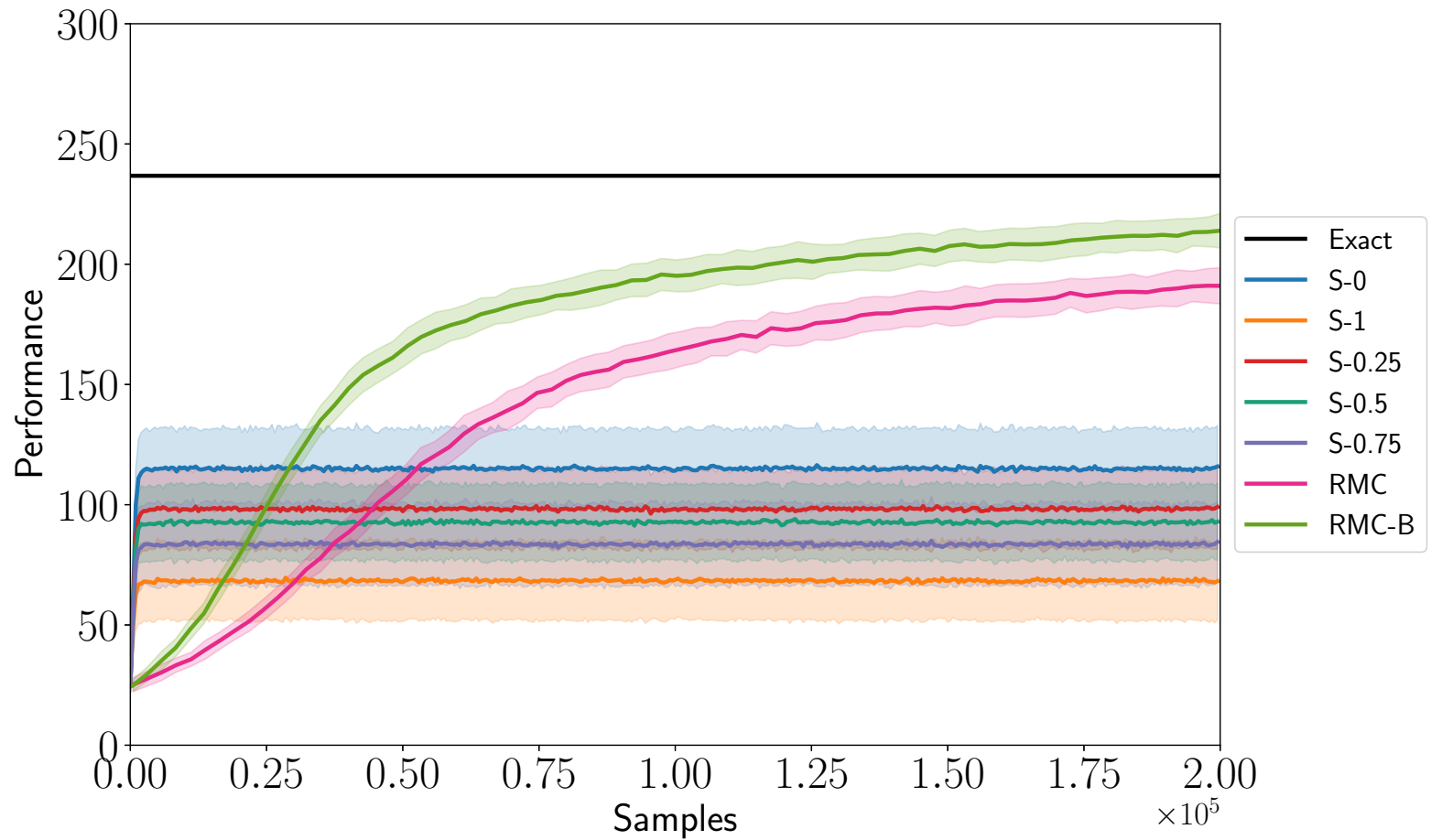
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Related work

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- Renewal theory for RL: [Marbach & Tsitsiklis 2001, 2003]
 - Average reward criterion
 - Relative value function for average reward

Limitation of RMC

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- ⊖ Renewal could take a long time

Limitation of RMC

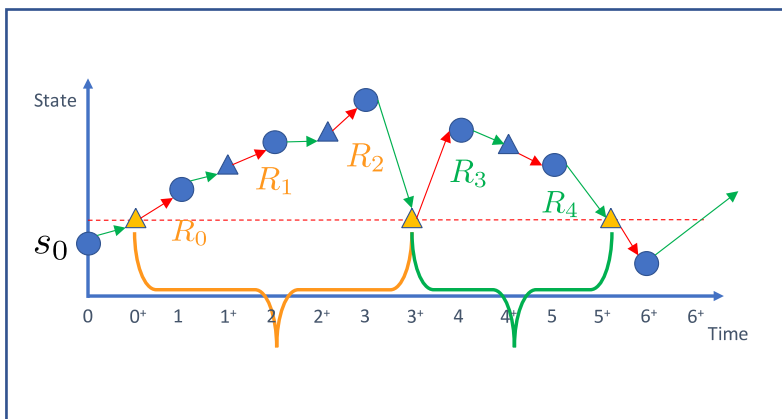
- ⊖ Renewal could take a long time
- ⊕ Two techniques to overcome this:

Limitation of RMC

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⊕ Two techniques to overcome this:

Post-decision state model

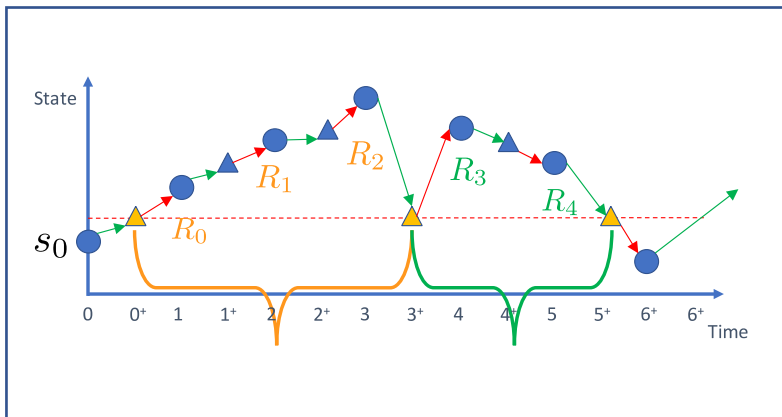


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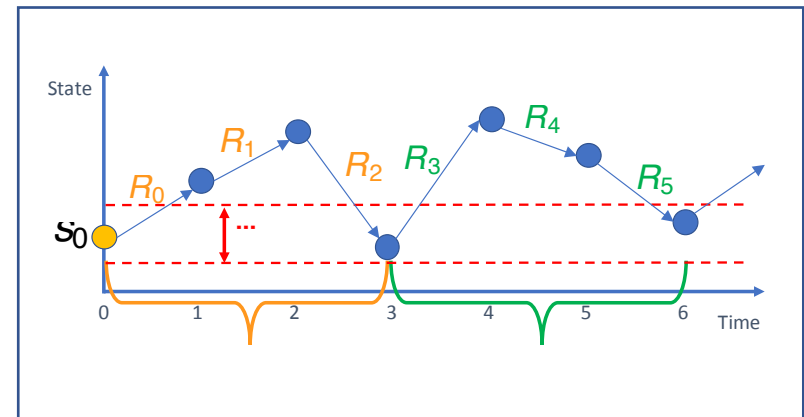
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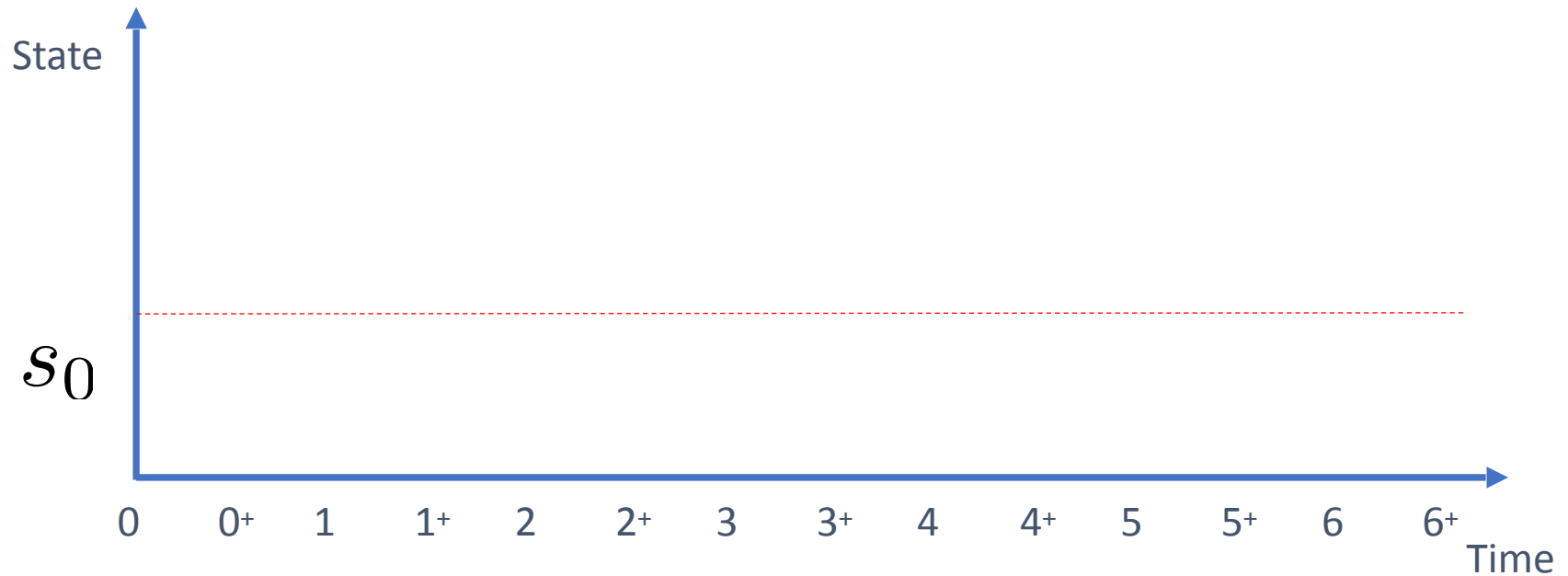
Post-decision state model



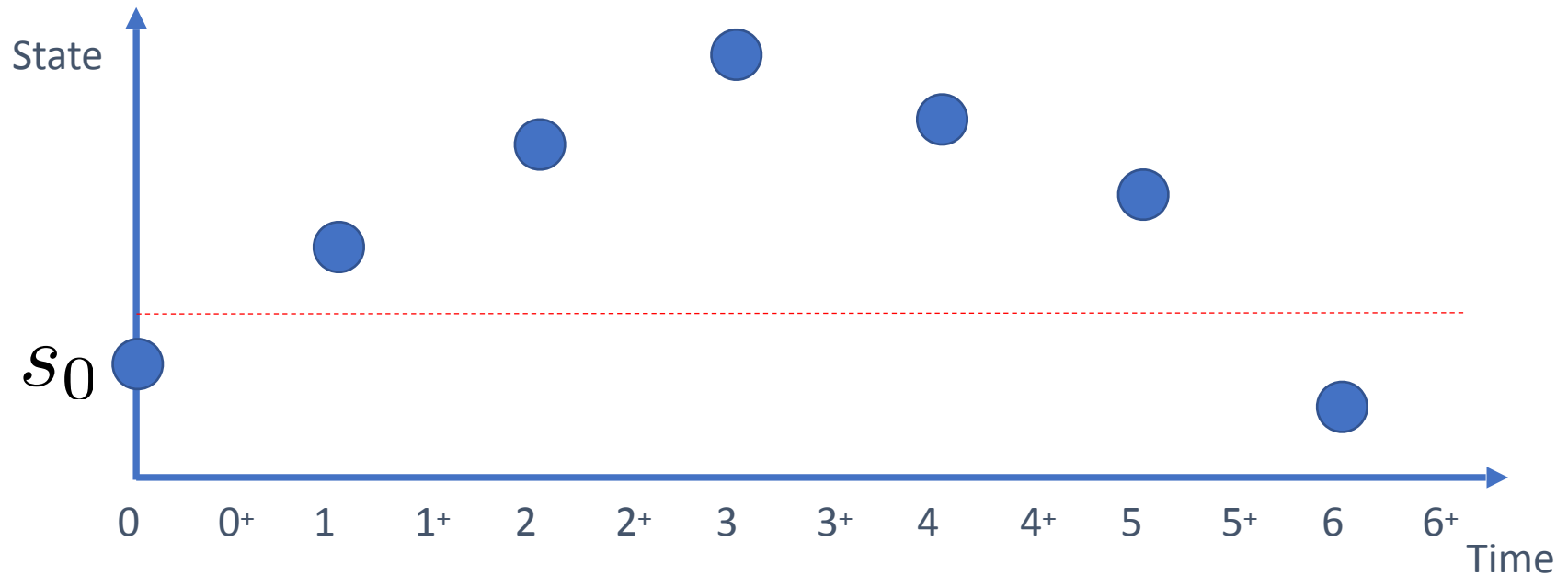
Approximate renewal model



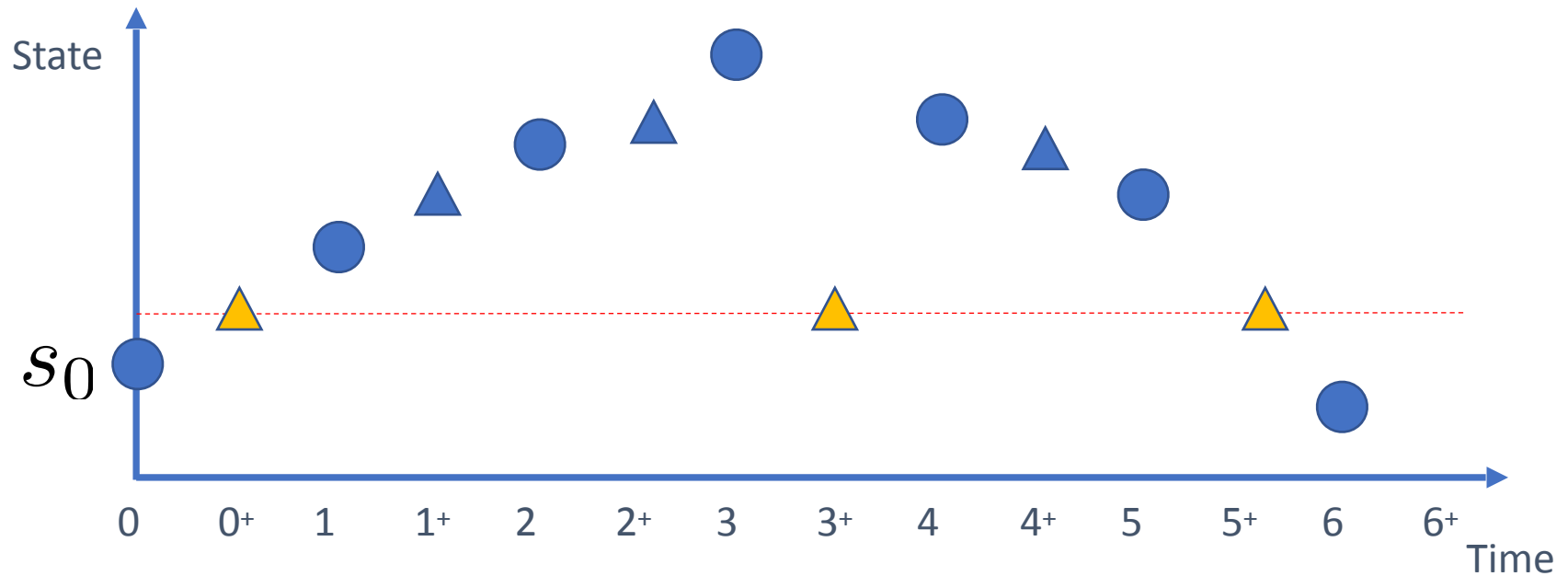
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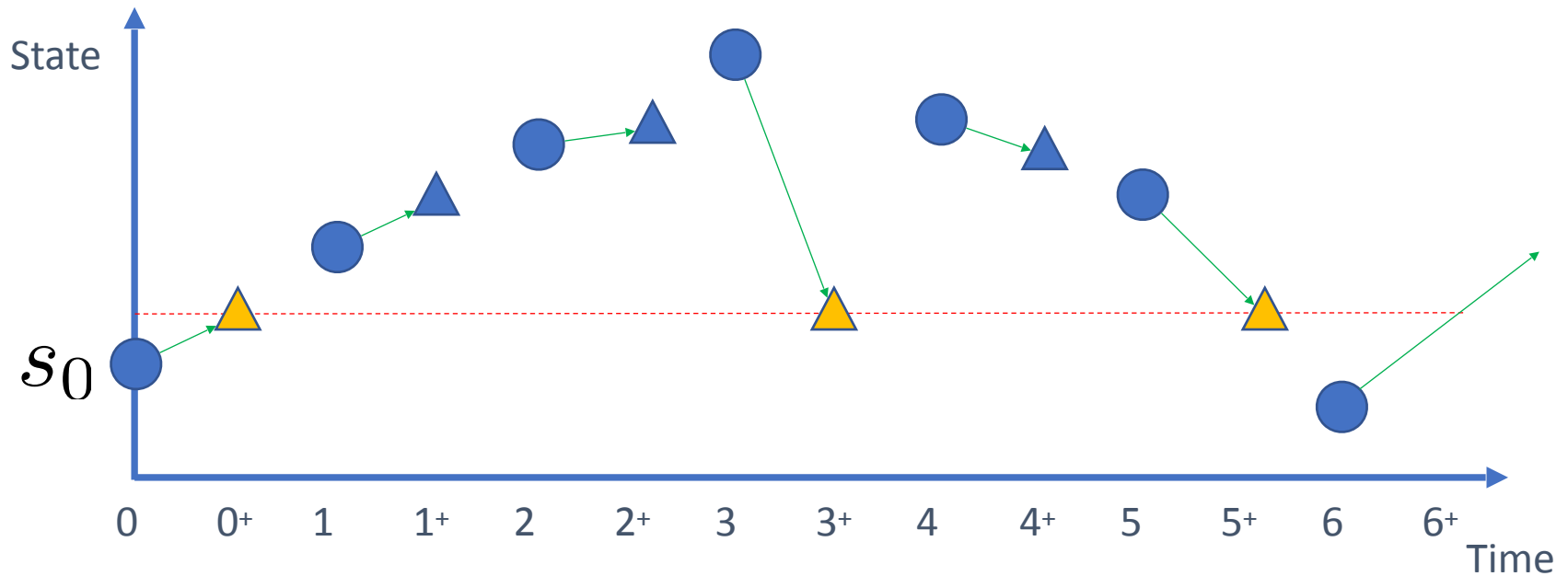
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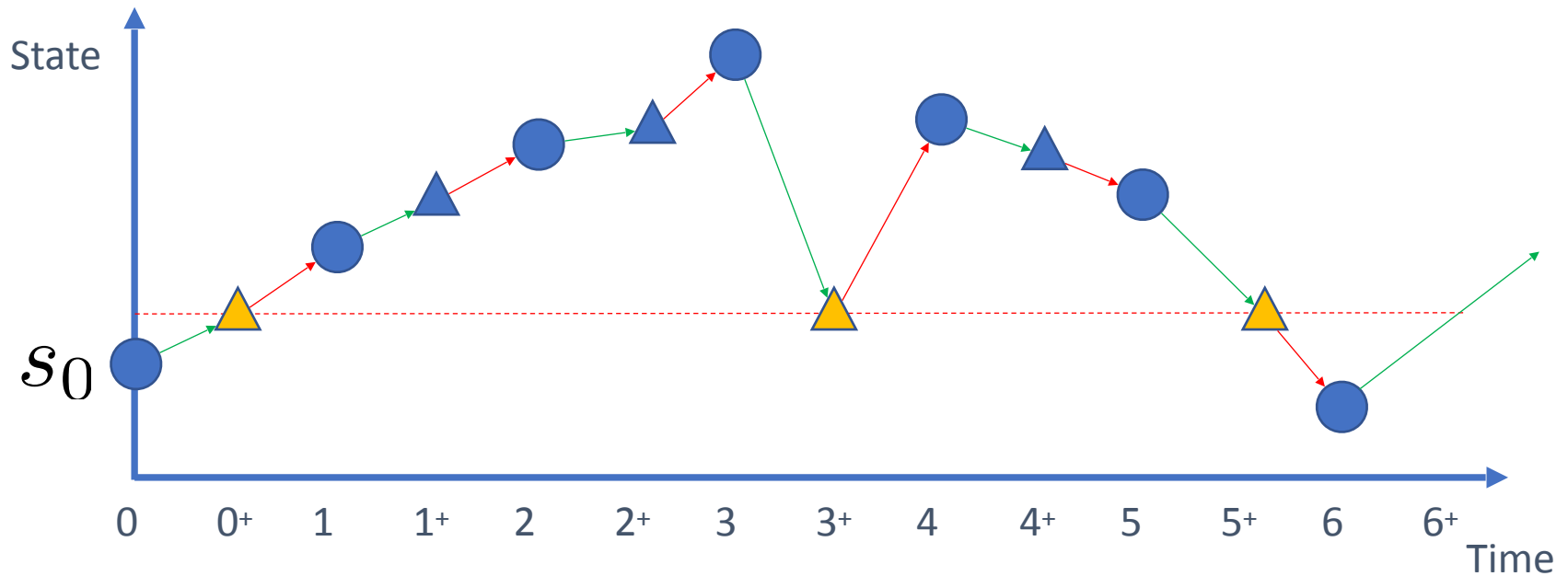
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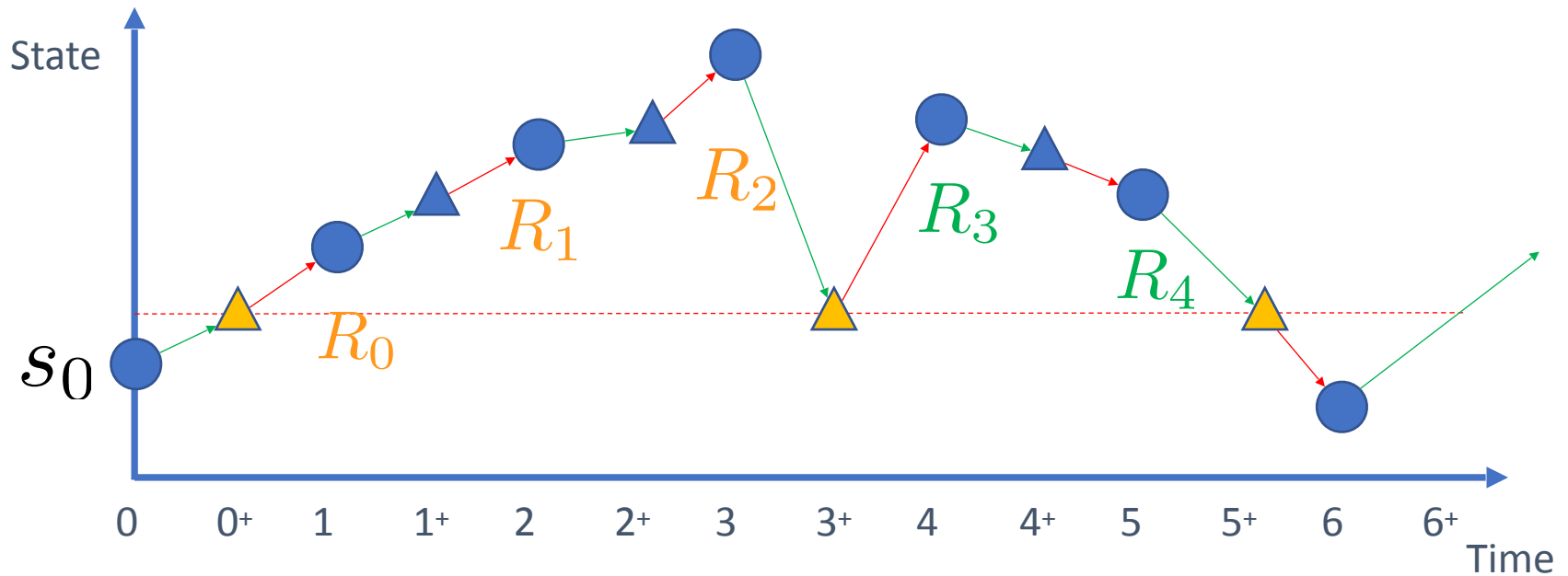
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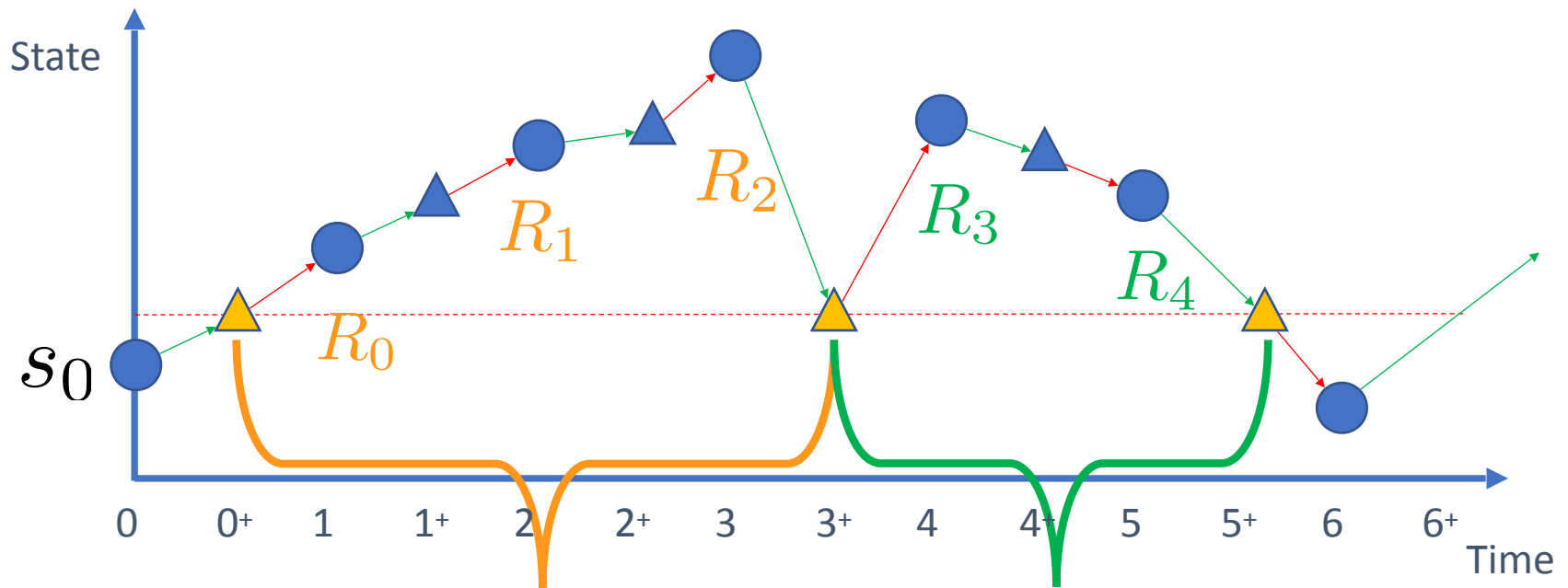
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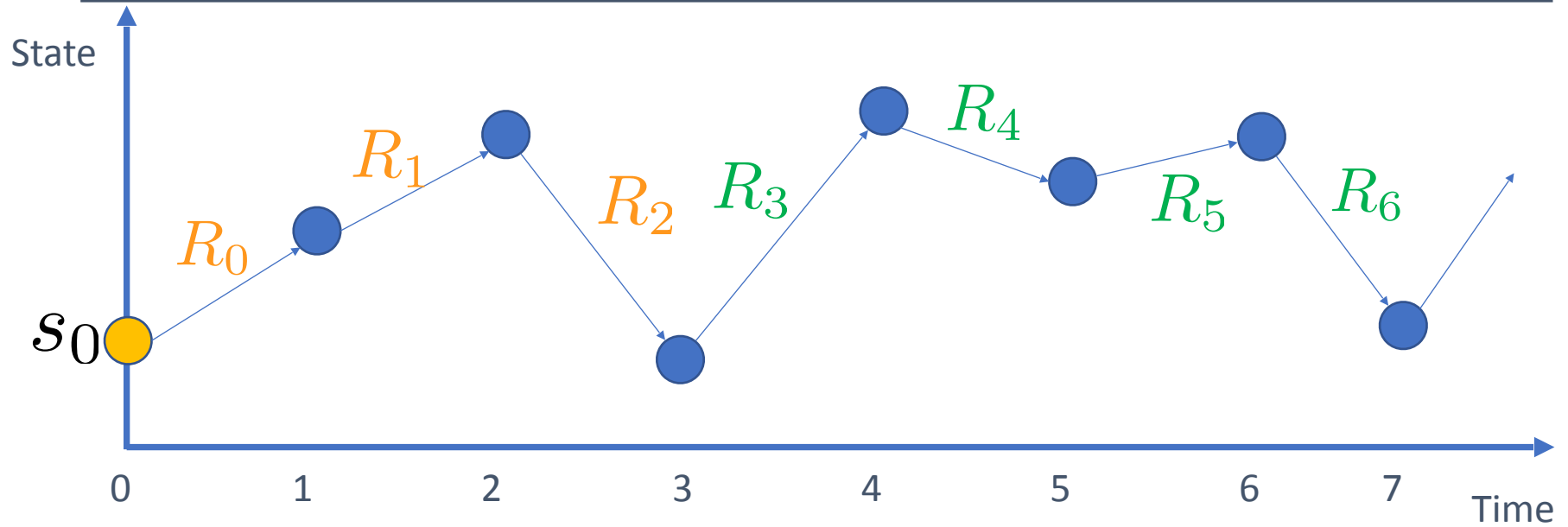


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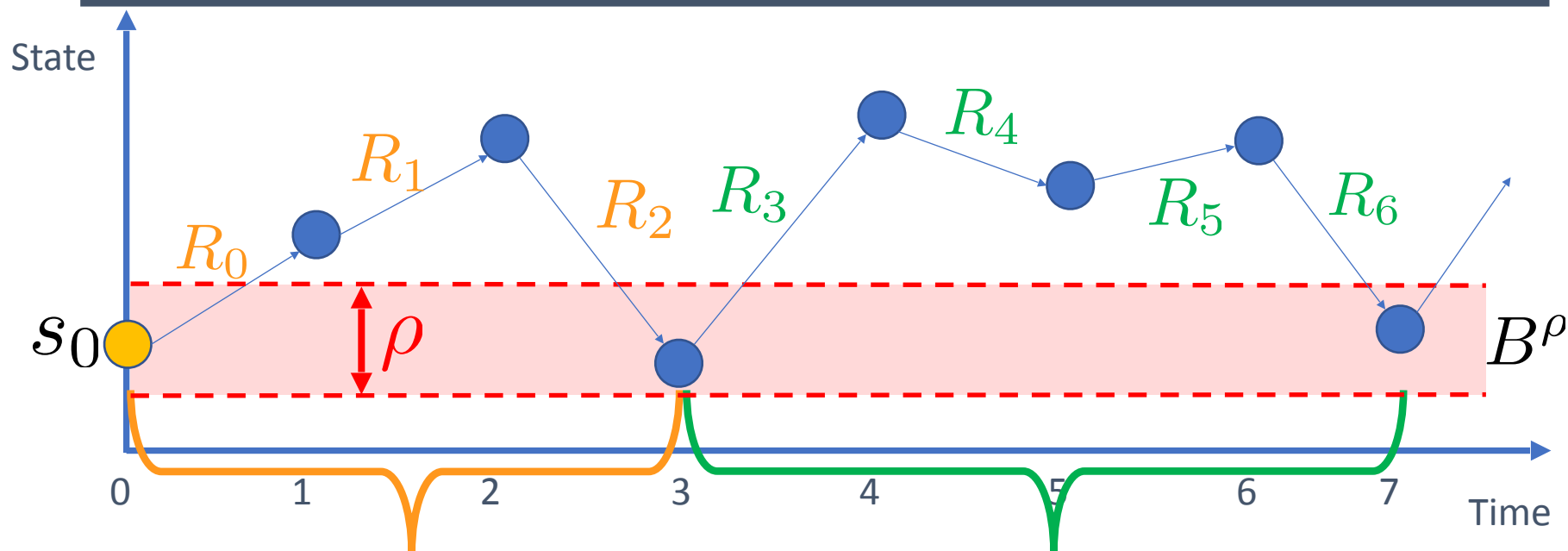


Renewals defined in terms of post-decision states

Approximate RMC

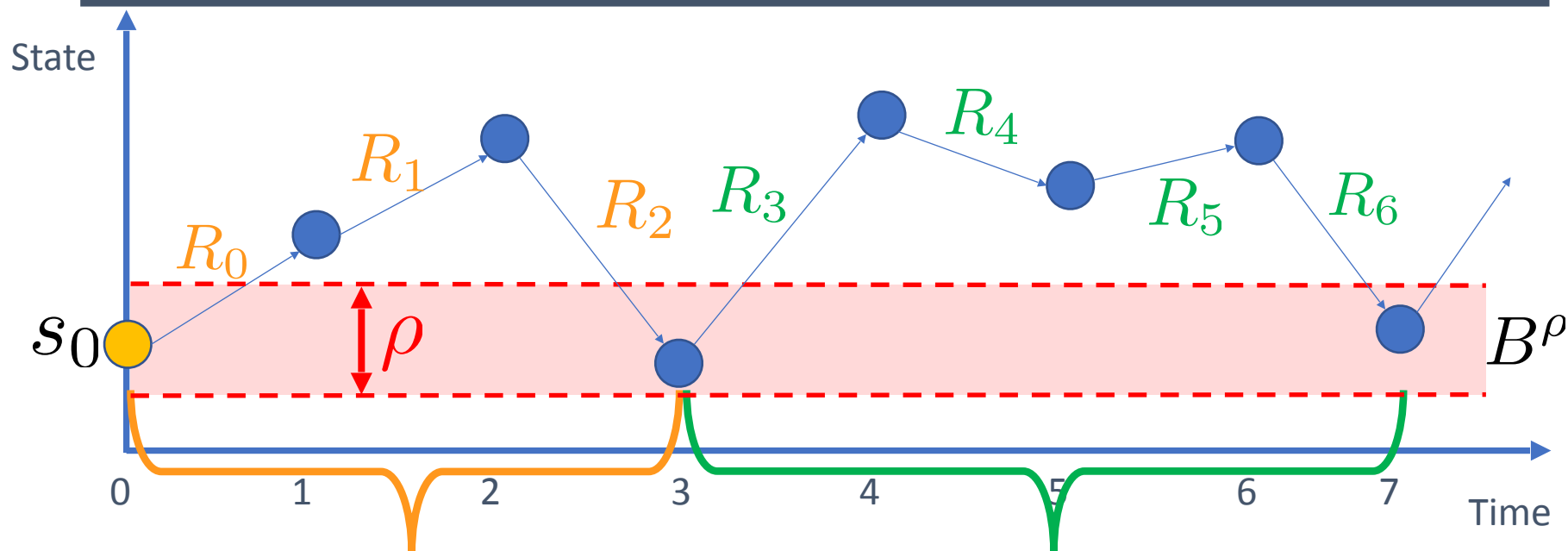


Approximate RMC



$$R^{\rho,(1)} = \sum_{t=0}^2 \gamma^t R_t, \quad T^{\rho,(1)} = \sum_{t=0}^2 \gamma^t \quad R^{\rho,(2)} = \sum_{t=3}^6 \gamma^{t-3} R_t, \quad T^{\rho,(2)} = \sum_{t=3}^6 \gamma^{t-3}$$

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$R^{(1)}, R^{(2)} \dots$ are i.i.d and $T^{(1)}, T^{(2)} \dots$ are i.i.d

$R_{\theta}^{\rho} = \mathbb{E}[R^{\rho,(n)}]$ and $T_{\theta}^{\rho} = \mathbb{E}[T^{\rho,(n)}]$ \triangleright estimated by $\hat{R}^{\rho}, \hat{T}^{\rho}$

Error bound

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V_θ is **Locally Lipschitz** in B^ρ

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$$|V_\theta(s) - V_\theta(s')| \leq L_\theta d_S(s, s')$$

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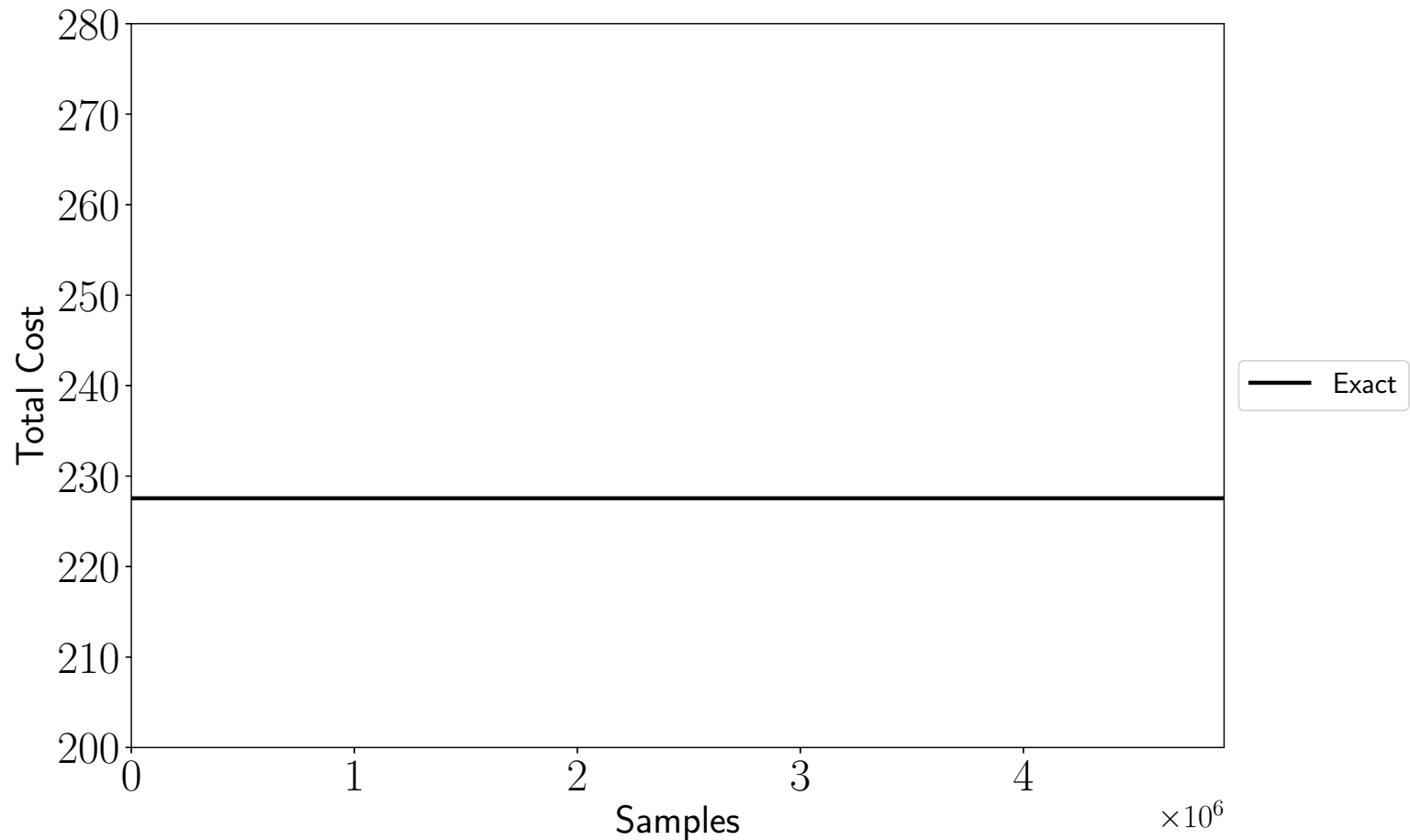
$$|V_\theta(s) - V_\theta(s')| \leq L_\theta d_S(s, s')$$



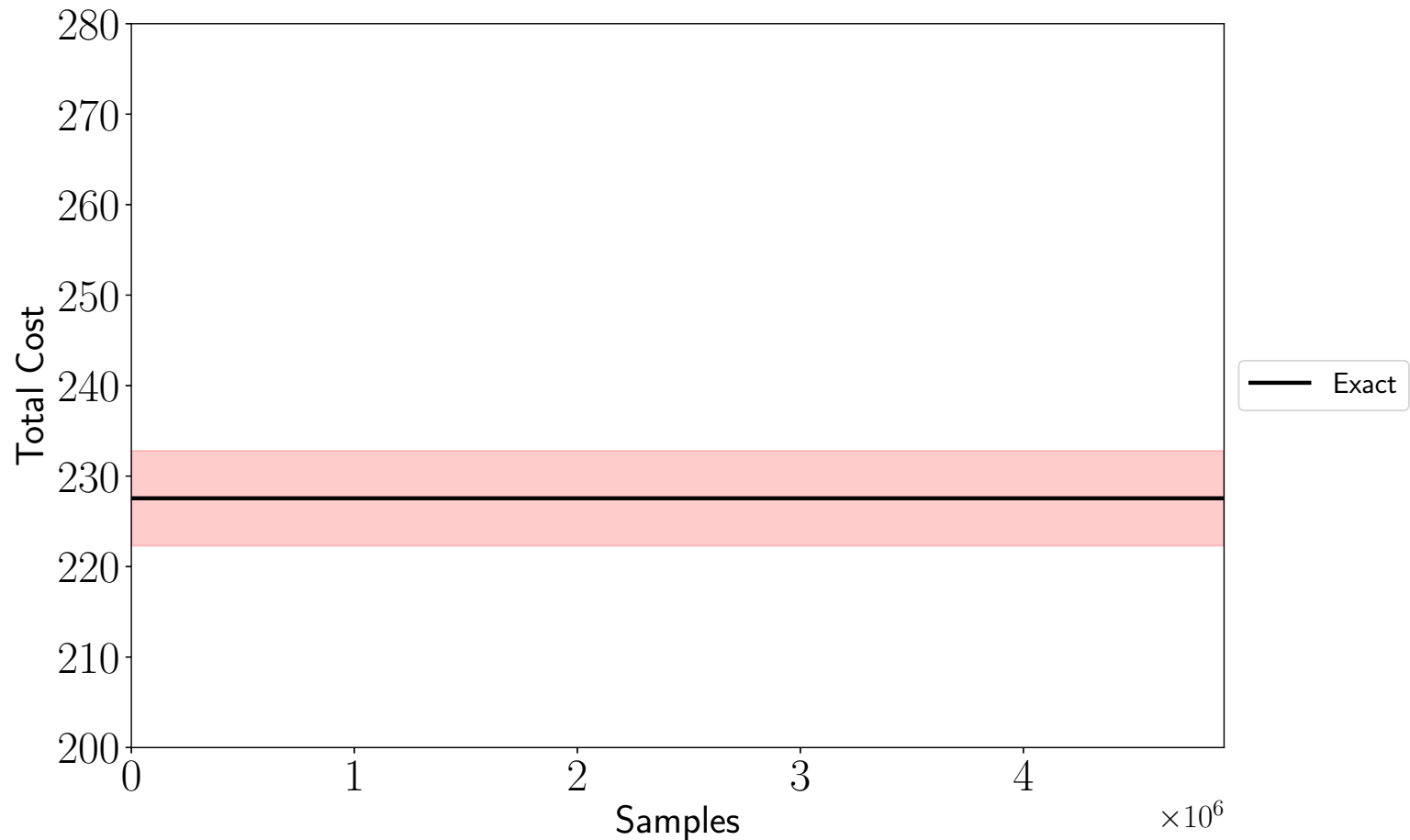
$$J_\theta - J_\theta^\rho \leq \dots \leq \frac{\gamma}{(1 - \gamma)} L_\theta \rho$$

Approximation error bounded by radius of approximation

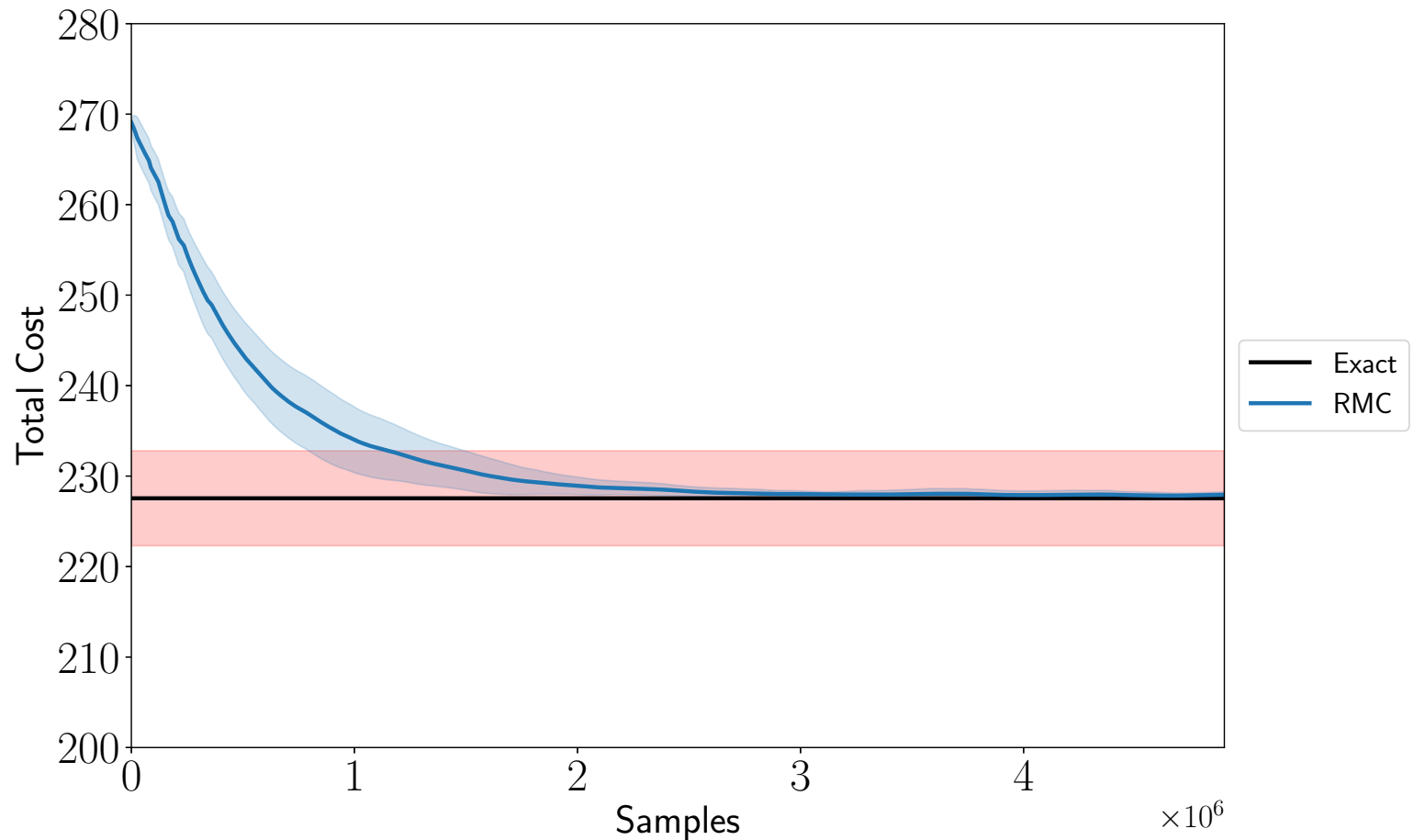
E.g. Inventory management



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- Not so useful in arbitrary high dimensional problems
- In high dimensional problems:
 - RMC can be used as a sub-component of main scheme
 - in the presence of hierarchies, can be used in a level with short renewals

Thank you