Structure of optimal strategies for remote estimation over Gilbert-Elliott channel with feedback

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June 27, 2017

Motivation

- Sequential transmission of data
- Zero delay in reconstruction

Motivation

Applications?

- Smart grids

Motivation

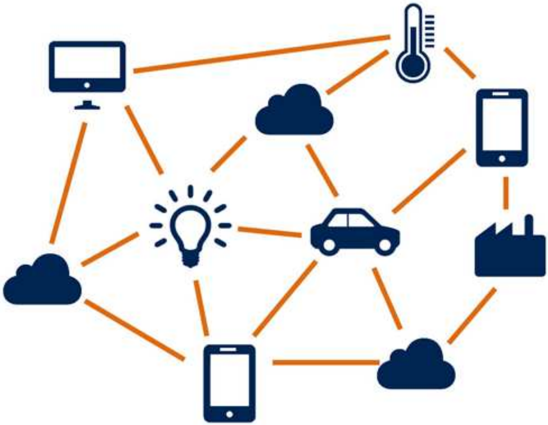
Applications?

- Environmental monitoring, sensor network

Motivation

Applications?

- Internet of things



Motivation

Applications?

- Smart grids
- Environmental monitoring, sensor network
- Internet of things

Salient features

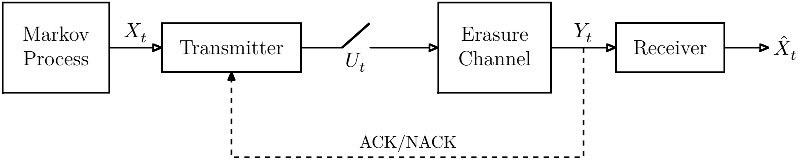
- Sensing is cheap
- Transmission is expensive
- Size of data-packet is not critical

Motivation

We study the structure of optimal strategies for a fundamental trade-off between estimation accuracy and transmission cost!

The model

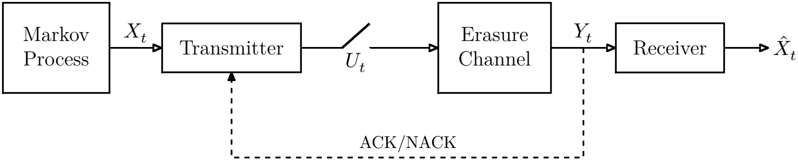
The remote-state estimation setup



Source model **Generic**: $X_t \in \mathbb{X}$, $\mathbb{X}$: finite or Borel-measurable;

Stylized: $X_{t+1} = aX_t + W_t$; $X_t \in \mathbb{X}$, W_t i.i.d.

The remote-state estimation setup



Transmitter $U_t = \mathbf{f_t}(X_{0:t}, S_{0:t-1}, Y_{0:t-1}) \in \{0, 1\}$

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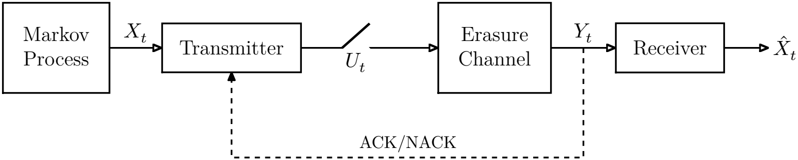

Channel model S_t Markovian; $S_t = 1$: channel ON, $S_t = 0$: channel OFF

State transition matrix Q.

$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \text{ and } S_t = 1 \\ \mathfrak{E}_1, & \text{if } U_t = 0 \text{ and } S_t = 1 \\ \mathfrak{E}_0, & \text{if } S_t = 0. \end{cases}$$

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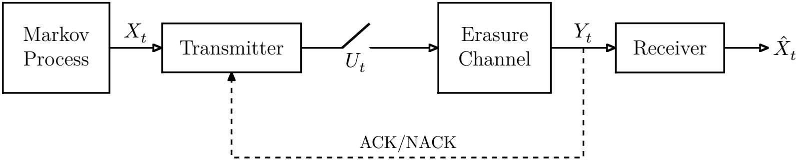

Receiver $\hat{X}_t = \mathbf{g_t}(Y_{0:t})$
Per-step distortion: $d(X_t - \hat{X}_t)$.
$d(\cdot)$: even and quasi-convex.

Communication strategies Transmission strategy $f = \{f_t\}_{t=0}^{\infty}$
Estimation strategy $g = \{g_t\}_{t=0}^{\infty}$

The infinite horizon optimization problem

Discounted setup: $\beta \in (0, 1)$

- $D_\beta(f, g) := (1-\beta)\mathbb{E}^{(f,g)}\left[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t) \,\middle|\, X_0 = 0\right]$
- $N_\beta(f, g) := (1-\beta)\mathbb{E}^{(f,g)}\left[\sum_{t=0}^{\infty} \beta^t U_t \,\middle|\, X_0 = 0\right]$

Long-term average setup: $\beta = 1$

- $D_1(f, g) := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{(f,g)}\left[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t) \,\middle|\, X_0 = 0\right]$
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The infinite horizon optimization problem

Problem

$$C^*_\beta(\lambda) := \inf_{(f,g)} D_\beta(f,g) + \lambda N_\beta(f,g),\ \beta \in (0,1]$$

The infinite horizon optimization problem

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Salient features

- Multiple decision makers — **Transmitter** and **Estimator**: decentralized control system
- Cooperative set-up — minimization of a **common objective function**
- Modeled as a **Team problem**; Team: Multiple decision makers to achieve a common goal

Decentralized control systems

Pioneers: Theory of teams

- Economics: Marschak, 1955; Radner, 1962
- Systems and control: Witsenhausen, 1971; Ho, Chu, 1972

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Remote-state estimation as Team problem

- **No packet drop** - Marshak, 1954; Kushner, 1964; Åstrom, Bernhardsson, 2002; Xu and Hespanha, 2004; Imer and Başar, 2005; Lipsa and Martins, 2011; Molin and Hirche, 2012; Nayyar, Başar, Teneketzis and Veeravalli, 2013; D. Shi, L. Shi and Chen, 2015
- **With packet drop** - Ren, Wu, Johansson, G. Shi and L. Shi, 2016; Chen, Wang, D. Shi and L. Shi, 2017;
- **With noise** - Gao, Akyol and Başar, 2015–2017

Structural results

Structure of optimal strategies

Generic model: $\mathbb{X}$ is finite or Borel-measurable.

Belief states based on common information

$$\pi_t^1(x) := \mathbb{P}^f(X_t = x \mid S_{0:t-1} = s_{0:t-1}, Y_{0:t-1} = y_{0:t-1}),$$
$$\pi_t^2(x) := \mathbb{P}^f(X_t = x \mid S_{0:t} = s_{0:t}, Y_{0:t} = y_{0:t}).$$

Theorem 1: structure of optimal strategies

$$U_t = f_t^*(X_t, S_{t-1}, \Pi_t^1),$$
$$\hat{X}_t = g_t^*(\Pi_t^2).$$

POMDP-like dynamic programming formulation.

Structure of optimal strategies

Stylized model: $X_{t+1} = aX_t + W_t$; W_t: Unimodal and symmetric.

Theorem 2: Optimal estimator

Time homogeneous!

$$\hat{X}_t = \begin{cases} Y_t, & \text{if } Y_t \notin \{\mathfrak{E}_0, \mathfrak{E}_1\}; \\ a\hat{X}_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\}. \end{cases}$$

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Theorem 2: Optimal transmitter

$X_t \in \mathbb{R}$; U_t is threshold based action:

$$U_t = \begin{cases} 1, & \text{if } |X_t - a\hat{X}_{t-1}| \geq k(S_{t-1}) \\ 0, & \text{if } |X_t - a\hat{X}_{t-1}| < k(S_{t-1}) \end{cases}$$

Proof sketch

Theorem 1

- Use notion of Irrelevant Information to show that $(X_t, S_{0:t-1}, Y_{0:t-1})$ is sufficient information at the transmitter
- Identify the common information $(S_{0:t-1}, Y_{0:t-1})$ at the transmitter and $(S_{0:t}, Y_{0:t})$ at the receiver
- Local information at the transmitter: X_t and at the receiver: $\varnothing$
- Belief states: at the transmitter $\pi_t^1 := \mathbb{P}(X_t \mid S_{0:t-1}, Y_{0:t-1})$, at the receiver $\pi_t^2 := \mathbb{P}(X_t \mid S_{0:t}, Y_{0:t})$
- **Common information approach - Nayyar, Mahajan, Teneketzis TAC'13**: show that (X_t, S_{t-1}, π_t^1) is sufficient statistic at the transmitter and π_t^2 is sufficient statistic at the receiver

Proof sketch

Theorem 2

- Change of variables: $E_t, E_t^+, \hat{E}_t$

$$Z_t = \begin{cases} aZ_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \notin \{\mathfrak{E}_0, \mathfrak{E}_1\} \end{cases}$$

- $E_t := X_t - aZ_{t-1}, \quad E_t^+ := X_t - Z_t, \quad \hat{E}_t := \hat{X}_t - Z_t$
- Step 1: **Forward induction method** utilizing majorization properties to show optimal $\hat{E}_t = 0$ — leads to the structure of optimal estimator
- Step 2: Fix the optimal estimator. Show **by constructing a threshold based prescription** that such a transmission strategy is optimal

Computation of optimal performances: autoregressive model

Step 1: computation of the performance of a threshold based strategy

$$f^{(k)}(E_t, S_{t-1}) = \begin{cases} 1, & \text{if } S_{t-1} = 0 \;\&\; |E_t| \geq k(S_{t-1}) \\ 0, & \text{if } S_{t-1} = 0 \;\&\; |E_t| < k(S_{t-1}). \end{cases}$$

$\tau^{(k)}$: the time a packet was last received successfully.

Step 1: computation of the performance of a threshold based strategy

$\tau^{(k)}$: the time a packet was last received successfully.

Till first successful reception

$$L_\beta^{(k)} := \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \;\Big|\; E_0 = 0, S_0 = 1\Big]$$

$$M_\beta^{(k)} := \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \;\Big|\; E_0 = 0, S_0 = 1\Big]$$

$$K_\beta^{(k)} := \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}} \beta^t U_t \;\Big|\; E_0 = 0, S_0 = 1\Big]$$

Step 1: computation of the performance of a threshold based strategy

E_t is regenerative process

Renewal relationships

$$D_\beta^{(k)} := D_\beta(f^{(k)}, g^*) = \frac{L_\beta^{(k)}}{M_\beta^{(k)}}$$

$$N_\beta^{(k)} := N_\beta(f^{(k)}, g^*) = \frac{K_\beta^{(k)}}{M_\beta^{(k)}}$$

Step 2: Optimality condition (JC & AM: TAC'17, NecSys '16)

$D_\beta^{(k)}$, $N_\beta^{(k)}$, $C_\beta^{(k)}$ - differentiable in k.

Theorem

If (k, λ) satisfies $\nabla_k D_\beta^{(k)} + \lambda \nabla_k N_\beta^{(k)} = \mathbf{0}$, then, $(f^{(k)}, g^*)$ optimal for costly comm. with cost λ.

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If (k, λ) satisfies $\nabla_k D_\beta^{(k)} + \lambda \nabla_k N_\beta^{(k)} = \mathbf{0}$, then, $(f^{(k)}, g^*)$ optimal for costly comm. with cost λ.

$C_\beta^*(\lambda) := C_\beta(f^{(k)}, g^*; \lambda)$ is continuous, increasing and concave in λ.

Step 2: Computation of optimal thresholds

Numerically compute $L_{\beta}^{(k)}$, $M_{\beta}^{(k)}$ and $K_{\beta}^{(k)}$; **Renewal relationship to compute** $C_{\beta}^{(k)}$.

Analytical formulae are difficult to obtain.

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Simulation based approach - JC, JS & AM ACC'17

- Two DP based approaches - Monte Carlo (MC) and Temporal Difference (TD)
  - MC - High variance due to one sample path; low bias
  - TD - Low variance due to *bootstrapping*; high bias

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- Two DP based approaches - Monte Carlo (MC) and Temporal Difference (TD)
  - MC - High variance due to one sample path; low bias
  - TD - Low variance due to *bootstrapping*; high bias
- Exploit regenerative property of the underlying state (error) process
- Renewal Monte Carlo (RMC) - low variance (independent sample paths from renewal) and low bias (since MC)

Step 2: Computation of optimal thresholds

Numerically compute $L_\beta^{(k)}$, $M_\beta^{(k)}$ and $K_\beta^{(k)}$; **Renewal relationship to compute** $C_\beta^{(k)}$.

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Key idea

- **Renewal Monte Carlo**
  - Pick a k, compute sample values L, M, K till first successful reception
  - Sample average to compute $L_\beta^{(k)}$, $M_\beta^{(k)}$, $K_\beta^{(k)}$.
- **Stochastic approximation** techniques to compute optimal k.

Step 2: Computation of optimal thresholds

Key steps of the algorithms

- **Noisy policy evaluation** - MC till successful reception: constitutes one episode; sample average over few episodes to find $\hat{L}$, $\hat{M}$, $\hat{K}$ and hence $\hat{C}$.

- **Policy improvement - Smoothed Functional**

$$\hat{k}_{i+1} = \hat{k}_i - \gamma_i \frac{\eta}{2\tilde{\beta}} \left(\hat{C}(\hat{k}_i + \tilde{\beta}\eta) - \hat{C}(\hat{k}_i - \tilde{\beta}\eta) \right)$$

- $k = [k(0), k(1)]^\intercal$; η : 2×1 Gaussian perturbation vector, $\tilde{\beta}$: tuning parameter

Smoothed Functional algo.- Katkovnik & Kulchitsky '72

- A **Simultaneous Perturbation** variant to estimate the gradient: $\nabla_k C_\beta^{(k)}$
- **Interpretation** - Cost function is convolved with a particular smooth kernel (e.g. Gaussian, Cauchy), effectively making the cost function more *convex-esque*
- Efficient scalability to higher dimensions

Simulation results to find optimal thresholds

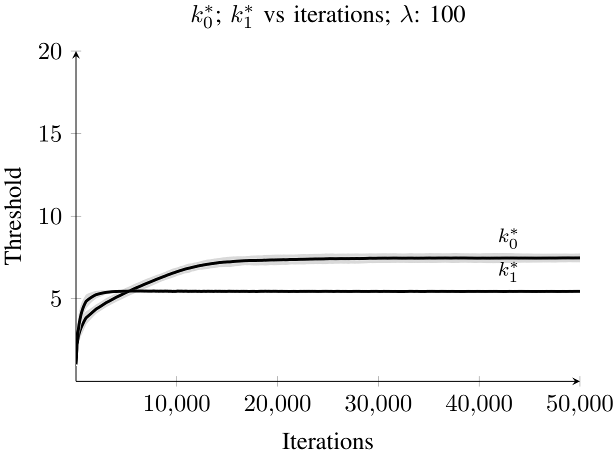

Figure: k_0^*, k_1^* plots for $\lambda = 100$: $\beta = 0.9$, $q_{00} = 0.3$, $q_{10} = 0.1$.

Simulation results to find optimal thresholds

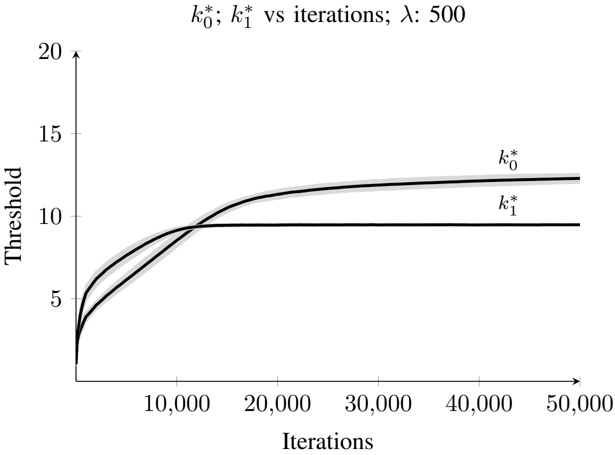

Figure: k_0^*, k_1^* plots for $\lambda = 500$: $\beta = 0.9$, $q_{00} = 0.3$, $q_{10} = 0.1$.

Optimal performance from simulation

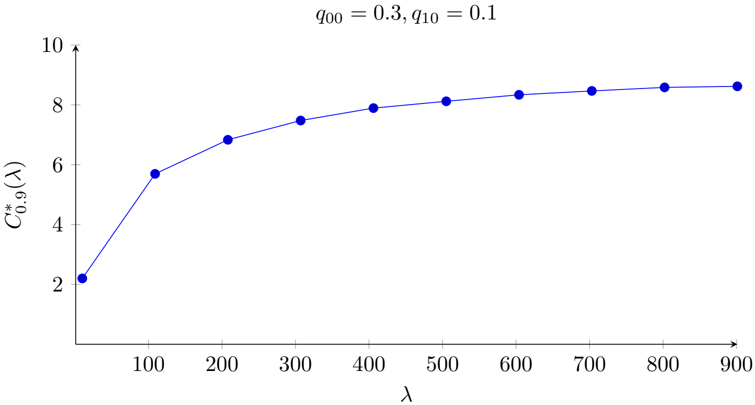

Figure: $C^*_{0.9}(\lambda)$ vs λ: $q_{00} = 0.3$, $q_{10} = 0.1$.

Future work

- Computation of the optimal constrained performance using stochastic approximation based method
- Extension of the results to vector valued source processes.

Thank you