Average cost optimal threshold strategies for remote state estimation with communication costs

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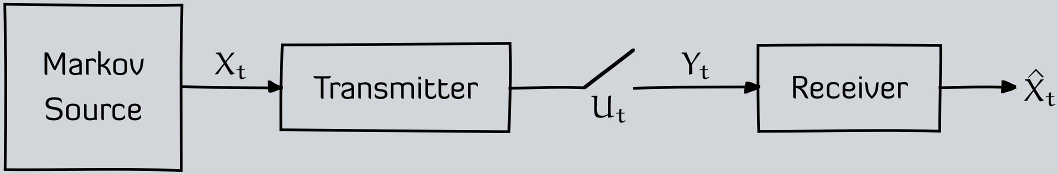
The communication system



Source
- $X_t \in \mathbb{Z}$
- Transition matrix P is Toeplitz, i.e., $P_{i,j} = p_{|i-j|}$, where $p_0 \geqslant p_1 \geqslant \cdots$.

Transmitter $U_t = f_t(X_{1:t}, U_{1:t-1})$ and $Y_t = \begin{cases} X_t & \text{if } U_t = 1 \\ \varepsilon & \text{if } U_t = 0 \end{cases}$

Receiver
- $\hat{X}_t = g_t(Y_{1:t})$
- Distortion: $d(X_t - \hat{X}_t)$ where $d(e) = d(-e) \leqslant d(e+1)$

Communication Strategies
- Transmission strategy $f = \{f_t\}_{t=0}^{\infty}$.
- Estimation strategy $g = \{g_t\}_{t=0}^{\infty}$.

The constrained optimization problem

$$\min_{(f,g)} D_\beta(f,g) \quad \text{such that } N_\beta(f,g) \leqslant \alpha$$

Minimize expected distortion such that expected # of transmissions is less than α

Discounted setup

$$D_\beta(f,g) = (1-\beta)\,\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$$

$$N_\beta(f,g) = (1-\beta)\,\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t \,\Big|\, X_0 = 0\Big]$$

Average cost setup

$$D_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$$

$$N_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\Big[\sum_{t=0}^{T-1} U_t \,\Big|\, X_0 = 0\Big]$$

Salient Features

Comparision to Information Theory

- As in information theory, the optimization problem may be viewed as minimizing average distortion under an average-power constraint.
- Unlike information theory, the source reconstruction must be done in real-time (or with zero delay).
- Therefore, classical information theory techniques do not work. Source-channel separation is not optimal.
- We use the decentralized control approach to real-time communication (following Witsenhausen, Walrand-Varaiya, Teneketzis, . . .)

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Comaprision to decentralized control

- Two decision makers—the transmitter and the receiver.
- (One-sided) nested information structure:

  the transmitter knows all the information available to the receiver.
- Constrained optimization problem, where the constraint does not depend on the "common information" (i.e., the information at the receiver).

Literature Overview

[Imer-Başar 2005 & 2010]

Fixed number of transmissions for finite horizon LQG setup.

[Lipsa-Martins 2009 & 2011, Molin-Hirche 2009]

Remote estimation with communication cost for finite horizon LQG setup.

[Nayyar-Başar-Teneketzis-Veeravalli 2013]

Remote estimation with communication cost for finite horizon Markov chain setup. Also considered energy harvesting at the transmitter.

A large literature on event-driven communication . . .

Assumptions on the model

(A0) $X_t \in \mathbb{Z}$, and $X_0 = 0$.

(A1) The transition matrix is Toeplitz with decaying off-diagonal terms.

$$P = \begin{bmatrix} \ddots & p_0 & \ddots & & & \\ \cdots & p_1 & p_0 & p_1 & \cdots & \\ & \ddots & p_1 & p_0 & p_1 & \cdots \\ & & \ddots & \ddots & p_0 & \ddots \end{bmatrix} \quad \text{and} \quad p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots$$

- Nayyar et al, assumed that the transistion matrix was banded, that is, $\exists b$ such that $p_k = 0$, for all $k \geqslant b$.

(A2) The distortion function is even and increasing on $\mathbb{Z}_{\geqslant 0}$.

$$\forall e \in \mathbb{Z}_{\geqslant 0} : \quad d(e) = d(-e) \quad \text{and} \quad d(e) \leqslant d(e+1).$$

Lagrange Relaxation

$$\min_{(f,g)} D_\beta(f, g) \quad \text{such that } N_\beta(f, g) \leqslant \alpha$$

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Lagrange Relaxation

$$C^*_\beta(\lambda) \coloneqq \inf_{(f,g)} C_\beta(f,g;\lambda) \quad \text{where } C_\beta(f,g;\lambda) = D_\beta(f,g) + \lambda N_\beta(f,g)$$

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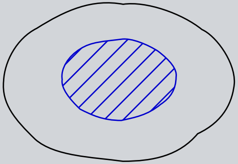

Search space of strategies (f, g)

- Restrict the search space of strategies (f, g) by identifying structure of optimal tranmission and estimation strategies.
- Difficulty: Non-classical information structure

Structure of optimal estimator (Nayyar et al, 2013)

Transmitted Process

Let Z_t denote the most recently transmitted value of the Markov source.

$$Z_0 = 0 \quad \text{and} \quad Z_t = \begin{cases} X_t & \text{if } U_t = 1; \\ Z_{t-1} & \text{if } U_t = 0. \end{cases}$$

The estimator can keep track of Z_t as follows:

$$Z_0 = 0 \quad \text{and} \quad Z_t = \begin{cases} Y_t & \text{if } Y_t \neq \varepsilon; \\ Z_{t-1} & \text{if } Y_t = \varepsilon. \end{cases}$$

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Theorem 1 The process $\{Z_t\}_{t=0}^{\infty}$ is a sufficient statistic at the estimator and an optimal estimation strategy is given by

$$\hat{X}_t = g_t^*(Z_t) = Z_t \qquad (\star)$$

Remark

- The optimal estimation strategy is time-homogeneous and can be specified in closed form.

Structure of optimal transmitter (Nayyar et al, 2013)

Error process Let $E_t = X_t - Z_{t-1}$ denote the error process. $\{E_t\}_{t=0}^{\infty}$ is a controlled Markov process where

$$E_0 = 0 \quad \text{and} \quad \mathbb{P}(E_{t+1} = n \mid E_t = e, U_t = u) = \begin{cases} P_{0n}, & \text{if } u = 1; \\ P_{en}, & \text{if } u = 0. \end{cases}$$

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Theorem 2 When the estimation strategy is of the form ($\star$), then $\{E_t\}_{t=0}^{\infty}$ is a sufficient statistic at the transmitter.

Furthermore, an optimal transmission strategy is characterized by a time-varying threshold $\{k_t\}_{t=0}^{\infty}$, i.e.,

$$U_t = f_t(E_t) = \begin{cases} 1 & \text{if } |E_t| \geqslant k_t; \\ 0 & \text{if } |E_t| < k_t. \end{cases}$$

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Proof idea
- The proof of [Nayyar et al, 2013] was based on some majorization inequalities of [Hajek et al, 2009] for distributions with finite support.
- We extend these inequalities to distributions over integers using results of [Wang-Woo-Madiman, 2014].

Infinite horizon setup (for Lagrange relaxation)

Main idea

- Based on Thm 1, restrict attention to time-homogeneous estimation strategy

$$\hat{X}_t = g^*_t(Z_t) = Z_t$$

- Consider the problem of finding the "best response" estimation strategy.

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- Centralized stochastic control problem with countable state space and unbounded cost.
- Standard MDP results apply under mild technical assumptions.

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- Centralized stochastic control problem with countable state space and unbounded cost.
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Assum (A3) For every $\lambda \geq 0$, there exists a function $w : \mathbb{Z} \to \mathbb{R}$ and postive and finite constants μ_1 and μ_2 such that for all $e \in \mathbb{Z}$, we have that

$$\max\{\lambda, d(e)\} \leq \mu_1 w(e)$$

$$\max\Big\{ \sum_{n=-\infty}^{\infty} P_{en} w(n), \sum_{n=-\infty}^{\infty} P_{0n} w(n) \Big\} \leq \mu_2 w(e).$$

Structure of optimal transmitter for infinite horizon

Structure Under assumption (A3), optimal transmission strategy is characterized by time-homogeneous threshold k, i.e.,

$$U_t = f(E_t) = \begin{cases} 1 & \text{if } |E_t| \geqslant k; \\ 0 & \text{if } |E_t| < k. \end{cases}$$

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Dynamic program For $\beta \in (0,1)$, the optimal strategy is determined by the unique fixed point of the following DP:

$$V_\beta(e;\lambda) = \min \Big\{ (1-\beta)\lambda + \beta \sum_{n=-\infty}^{\infty} P_{0n} V_\beta(n;\lambda), \quad \text{Transmit}$$

$$(1-\beta)d(e) + \beta \sum_{n=-\infty}^{\infty} P_{en} V_\beta(n;\lambda) \Big\} \quad \text{Don't Transmit}$$

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Lagrange relaxation Let $f^*_\beta(\cdot; \lambda)$ be the time-homogeneous optimal transmission strategy.

$$C^*_\beta(\lambda) \coloneqq \inf_{(f,g)} C_\beta(f, g; \lambda) = C_\beta(f^*_\beta, g^*; \lambda) = V_\beta(0; \lambda)$$

The SEN Conditions and the long-term average setup

SEN Conditions

For any $\lambda \geqslant 0$, the value function $V_\beta(\cdot;\lambda)$ satisfy the SEN condition:

(S1) There exists a reference state $e_0 \in \mathbb{Z}$ such that $V_\beta(e_0;\lambda) < \infty$ for all $\beta \in (0,1)$.

(S2) Define $h_\beta(e;\lambda) = (1-\beta)^{-1}[V_\beta(e;\lambda) - V_\beta(e_0;\lambda)]$. There exists a function $K_\lambda : \mathbb{Z} \to \mathbb{R}$ such that $h_\beta(e;\lambda) \leqslant K_\lambda(e)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

(S3) There exists a non-negative (finite) constant L_λ such that $-L_\lambda \leqslant h_\beta(e;\lambda)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

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(S3) There exists a non-negative (finite) constant L_λ such that $-L_\lambda \leqslant h_\beta(e;\lambda)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

Vanishing discount approach

Let $f_1^*(\cdot;\lambda)$ be any limit point of $f_\beta^*(\cdot;\lambda)$ as $\beta \uparrow 1$.
Then the time-homogeneous transmission strategy $f_1^*(\cdot;\lambda)$ is optimal for $\beta = 1$ (the long-term average setup).

Furthermore, the performance of this optimal strategy is

$$C_1^*(\lambda) \coloneqq \inf_{(f,g)} C_1(f,g;\lambda) = C_1(f_1^*, g^*;\lambda) = \lim_{\beta\uparrow 1} V_\beta(0;\lambda) = \lim_{\beta\uparrow 1} C_\beta^*(\lambda).$$

Performance of a threshold based strategy

Threshold-based strategy

We analyze the performace of $(f^{(k)}, g^*)$, where

$$f^{(k)}(e) \coloneqq \begin{cases} 1, & \text{if } |e| \geqslant k; \\ 0, & \text{if } |e| < k. \end{cases}$$

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Cost until first transmission

Define $S^{(k)} = \{e \in \mathbb{Z} : |e| \leqslant k - 1\}$ and let $\tau^{(k)}$ be the stopping time when the Markov process starting at state 0 at time $t = 0$ escapes the set $S^{(k)}$.

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Define
$$L_\beta^{(k)} \coloneqq \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = 0\Big]$$

$$M_\beta^{(k)} \coloneqq \frac{1 - \mathbb{E}[\beta^{\tau^{(k)}} \mid E_0 = 0]}{1-\beta}$$

and

$$L_1^{(k)} \coloneqq \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} d(E_t) \Big| E_0 = 0\Big]$$

$$M_1^{(k)} \coloneqq \mathbb{E}[\tau^{(k)} - 1 \mid E_0 = 0]$$

Performance of a threshold based strategy (cont.)

Renewal relationships

$$D_\beta^{(k)} \coloneqq D_\beta(f^{(k)}, g^*) = \frac{L_\beta^{(k)}}{M_\beta^{(k)}}$$

$$N_\beta^{(k)} \coloneqq N_\beta(f^{(k)}, g^*) = \frac{1}{M_\beta^{(k)}} - (1 - \beta)$$

Performance of a threshold based strategy (cont.)

Renewal relationships

$$D^{(k)}_\beta \coloneqq D_\beta(f^{(k)}, g^*) = \frac{L^{(k)}_\beta}{M^{(k)}_\beta}$$

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Vanishing discount relationships

$$L^{(k)}_1 = \lim_{\beta\uparrow 1} L^{(k)}_\beta, \quad M^{(k)}_1 = \lim_{\beta\uparrow 1} M^{(k)}_\beta.$$

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$$D^{(k)}_1 = \lim_{\beta\uparrow 1} D^{(k)}_\beta = \frac{L^{(k)}_1}{M^{(k)}_1}$$

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Performance of a threshold based strategy: Computations

Analytic expressions for performace

Let $P^{(k)}$ and $Q^{(k)}_\beta$ be square matrices and $d^{(k)}$ is a column vector indexed by $S^{(k)}$ defined as follows:

$$P^{(k)}_{ij} \coloneqq P_{ij}, \quad \forall i, j \in S^{(k)},$$

$$Q^{(k)}_\beta \coloneqq [I_{2k-1} - \beta P^{(k)}]^{-1},$$

$$d^{(k)} \coloneqq [d(-k+1), \dots, d(k-1)]^\top$$

Then,

$$L^{(k)}_\beta = [Q^{(k)}_\beta]_0 d^{(k)} \quad \text{and} \quad M^{(k)}_\beta = [Q^{(k)}_\beta]_0 \mathbf{1}_{2k-1}.$$

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$D^{(k)}_\beta$ and $N^{(k)}_\beta$ can be computed using these expressions.

Optimal stategy for the Lagrange relaxation

Some inequalities $$L_\beta^{(k)} < L_\beta^{(k+1)}, \quad M_\beta^{(k)} < M_\beta^{(k+1)}, \quad D_\beta^{(k)} < D_\beta^{(k+1)}.$$

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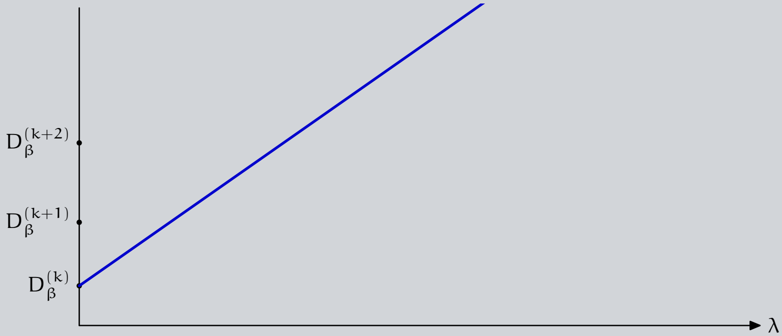

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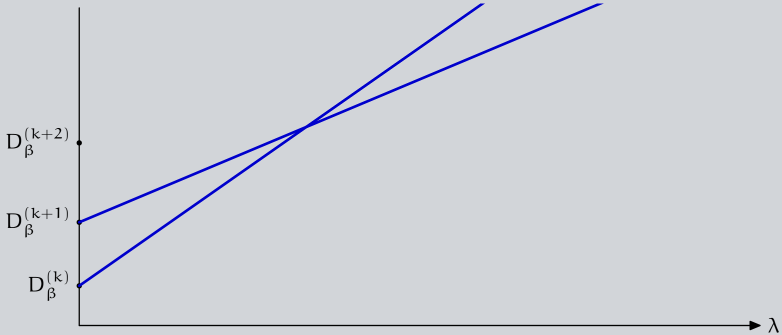

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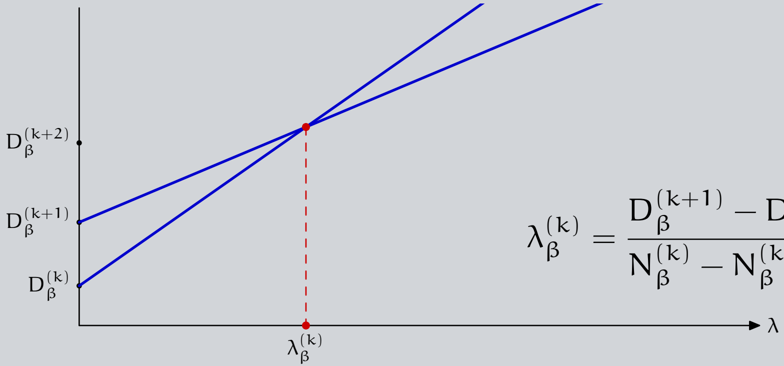

$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

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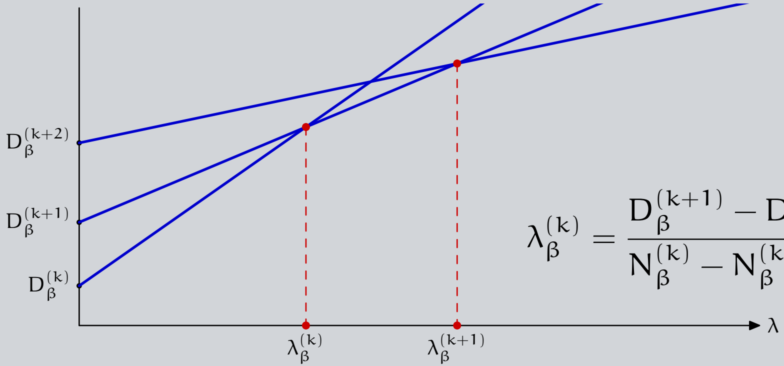

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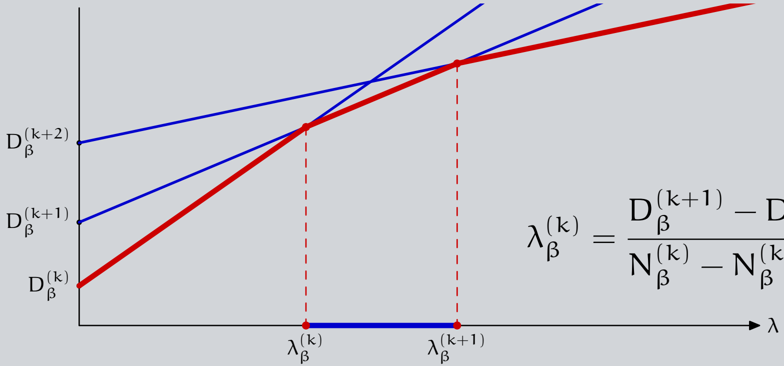

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Optimal performance

- For all $\lambda \in (\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}]$ the threshold strategy $f^{(k+1)}$ is optimal.
- $C_\beta^*(\lambda) = \min_{k \in \mathbb{Z}} C_\beta^{(k)}$ is piecewise linear, continuous, concave, and increasing function of λ.

Back to the constrained optimization problem

Bernoulli randomized strategy

Let $\theta \in [0, 1]$ and f_1 and f_2 be two stationary strategies.
The Bernoulli randomized strategy (f_1, f_2, θ) randomizes between f_1 and f_2 at each stage, choosing f_1 with probability θ and f_2 with probability $(1 - \theta)$.

Simple rand. strategy

A Bernoulli randomized strategy (f_1, f_2, θ) is simple if the actions prescribed by f_1 and f_2 differ only at one state.

Main result

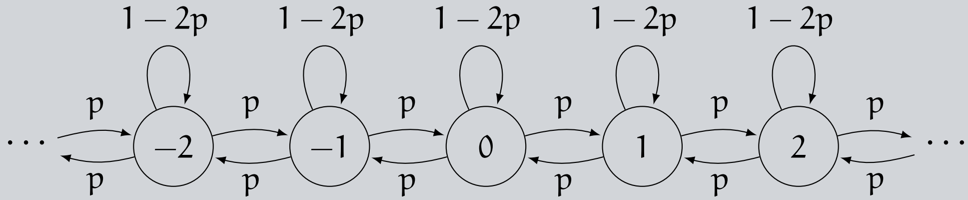
Define $k^*_\beta = \sup\{k \in \mathbb{Z}_{\geqslant 0} : N^{(k)}_\beta \geqslant \alpha\}$ and let θ be such that

$$\theta N^{(k^*_\beta)}_\beta + (1 - \theta) N^{(k^*_\beta + 1)}_\beta = \alpha$$

Then, the Bernoulli simple randomized strategy $(f^{(k^*_\beta)}, f^{(k^*_\beta + 1)}, \theta)$ is optimal for the constrained optimization problem for $\beta \in (0, 1]$.

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$



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Discounted cost Let $K_\beta = -2 - (1-\beta)/\beta p$ and $m_\beta = \cosh^{-1}(-K_\beta/2)$.

$$D_\beta^{(k)} = \frac{\sinh(k m_\beta) - k \sinh(m_\beta)}{2\sinh^2(k m_\beta/2)\sinh(m_\beta)}$$

$$N_\beta^{(k)} = \frac{2\beta p \sinh^2(m_\beta/2)\cosh(k m_\beta)}{\sinh^2(k m_\beta/2)} - (1-\beta)$$

Average cost $D_1^{(k)} = \dfrac{k^2-1}{3k}$ and $N_1^{(k)} = \dfrac{2p}{k^2}$

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

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$\lambda_\beta^{(k)}$ can be computed in terms of $D_\beta^{(k)}$ and $N_\beta^{(k)}$.

Average cost $D_1^{(k)} = \dfrac{k^2-1}{3k}$ and $N_1^{(k)} = \dfrac{2p}{k^2}$

$$\lambda_1^{(k)} = \frac{k(k+1)(k^2+k+1)}{6p(2k+1)}$$

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

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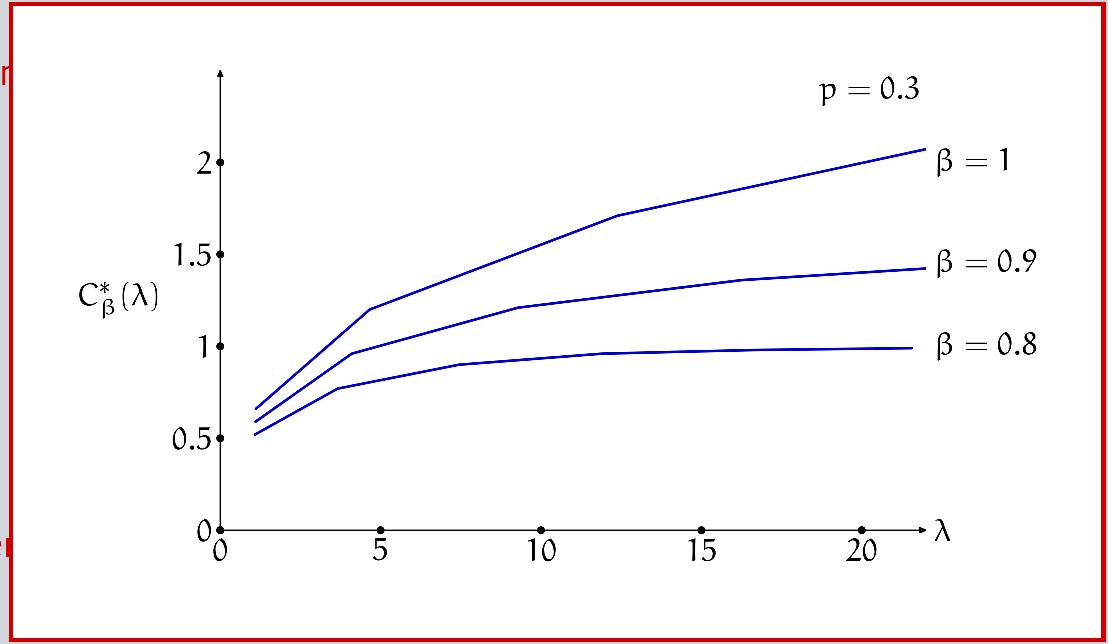

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An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

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$$k_\beta^* = \sup \left\{ k \in \mathbb{Z}_{\geqslant 0} : \frac{\sinh^2(m_\beta/2) \cosh(k m_\beta)}{\sinh^2(k m_\beta/2)} \geqslant \frac{1+\alpha-\beta}{2\beta p} \right\}$$

Average cost $D_1^{(k)} = \dfrac{k^2-1}{3k}$ and $N_1^{(k)} = \dfrac{2p}{k^2}$

$$k_1^* = \left\lfloor \sqrt{\frac{2p}{\alpha}} \right\rfloor$$

An example: Symmetric birth-death Markov Chain

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$

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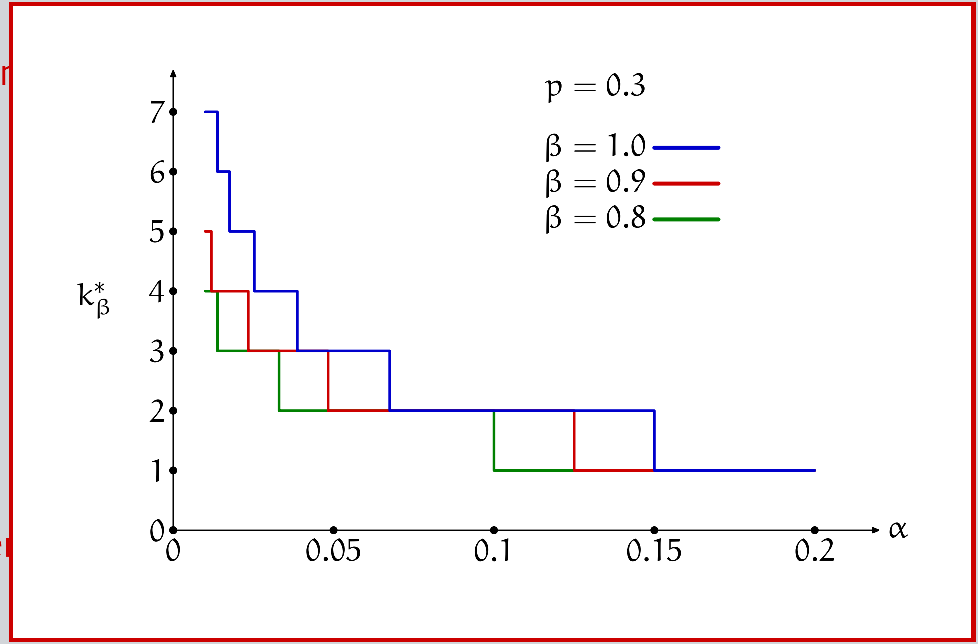

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Summary and Conclusion

Problem formulation

- Real-time transmission of a Markov source under constraints on the number of transmissions.
- Investigated both discounted and average cost infinite horizon setups.
- Modeled as a decentralized stochastic control problem with two decision maker.
- As long as the transmitter uses a symmetric threshold based strategy, the estimation strategy does not depend on the transmission strategy.
- The problem of find the "best response" transmitter is a centralized stochastic control problem.

Main results

- Simple Bernoulli randomized strategies $(f^{(k^*)}, f^{(k^*+1)}, \theta)$ are optimal.
- k^* and θ can be computed easily.