Optimal sequential transmission over broadcast channel with nested feedback

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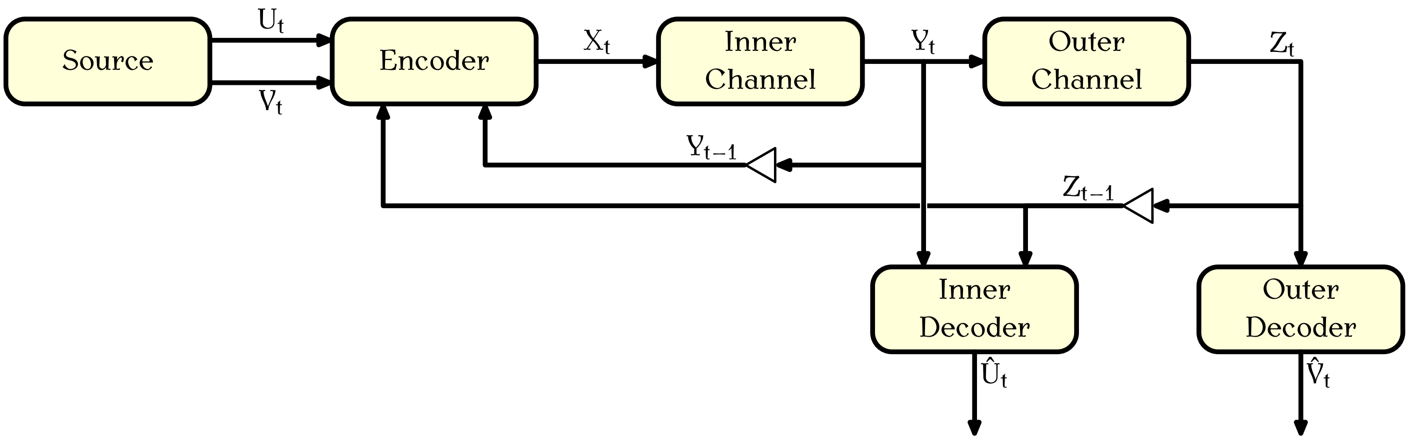
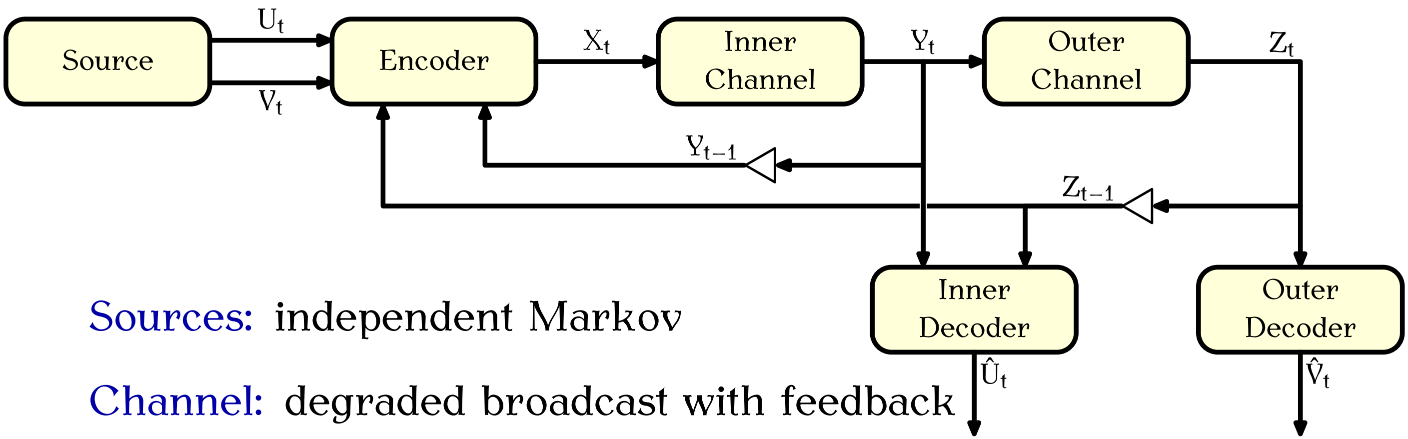
Model



Source
U_t
V_t
Encoder
X_t
Inner Channel
Y_t
Outer Channel
Z_t
Y_{t-1}
Z_{t-1}
Inner Decoder
Outer Decoder
$\hat{U}_t$
$\hat{V}_t$

Model



Sources: independent Markov

Channel: degraded broadcast with feedback

Encoder: $X_t = c_t(U^t,\ V^t,\ X^{t-1},\ Y^{t-1},\ Z^{t-1})$

Inner Decoder: $\hat{U}_t = g_{1,t}(Y^t,\ Z^{t-1})$

Outer Decoder: $\hat{V}_t = g_{2,t}(Z^t)$

Total distortion: $\mathbb{E}\left\{\sum_{t=1}^{T} \rho_{1,t}(U_t,\ \hat{U}_t) + \rho_{2,t}(V_t,\ \hat{V}_t)\right\}.$

Aim

Obtain structural results and sequential decomposition

Similar in spirit to the results in point-to-point communication.

- No feedback (Teneketzis, TIT 2006, M and Teneketzis, TIT 2009)
- Noiseless feedback (Walrand and Varaiya, TIT 1982)
- Noisy feedback (M and Teneketzis, JSAC 2008)

Why consider structural results?

The domain of encoding and decoding functions increases with time.

- Difficult to find optimal strategies
- Difficult to implement a strategy

Want a sufficient statistic that does not increase with time.

Structural results can be useful for deriving additional results.

- For real-time communication over symmetric channels, uncoded transmission is optimal (Walrand and Varaiya, 1982)

Results also applicable to info theoretic setup

If we assume that the source is constant,

and $\rho_{i,t} \equiv 0$ for $t = 0, \ldots, T-1$ and $\rho_{i,T}(w, \hat{w}) = \begin{cases} 0, & \text{if } w = \hat{w}; \\ 1, & \text{otherwise.} \end{cases}$

then the setup is same as classical information theory setup.

For degraded broadcast channels, feedback does not improve capacity

(Leighton and Tan 1977, ElGamal 1978)

Feedback can help simplify communication scheme.

Motivated by recent results of Shayevitz and Feder (ISIT 07, 08). In particular, by Coleman's (ISIT 09) interpretation of the results.

Help in crystallizing ideas about general decentralized systems

The main idea of Walrand and Varaiya (1982) can be generalized to

- Information structures with shared information (M, Nayyar, and Teneketzis, 2008)
- delay-sharing information structures (Nayyar, M, and Teneketzis, ACC 2010)

The main idea of M and Teneketzis (2008, 2009) can be generalized to

- Arbitrary two agent team problems (M, 2008)

Solution Approach

Enc: $(U^t,\ V^t,\ X^{t-1},\ Y^{t-1},\ Z^{t-1})$ Inner dec: (Y^t, Z^{t-1}) Outer dec: (Z^t)

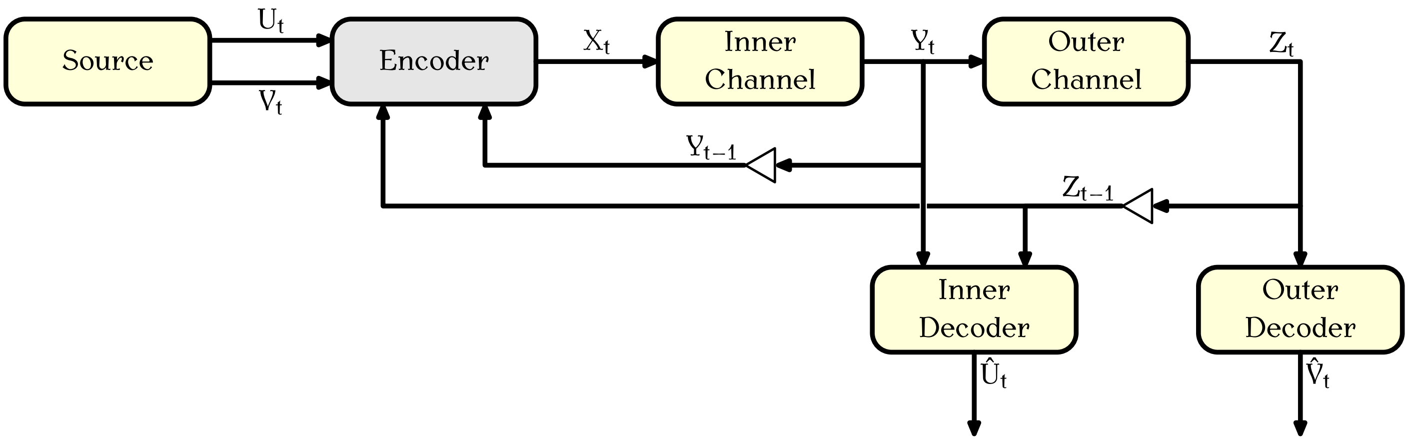
Step 1. $(U^{t-1}, V^{t-1}, X^{t-1})$ at the encoder can be ignored.

Step 2. Y^{t-1} at the encoder and the inner decoder can be compressed into $\Xi_{t-1} = \Pr(U_{t-1},\ V_{t-1} \mid Y^{t-1},\ Z^{t-1})$

Step 3. Z^{t-1} at all three can be compressed into $\Pi_{t-1} = \Pr(U_t,\ V_t,\ \Xi_{t-1} | Z^{t-1})$.

Step 4. Decoding is a filtration.

Step 1: Ignore past at encoder



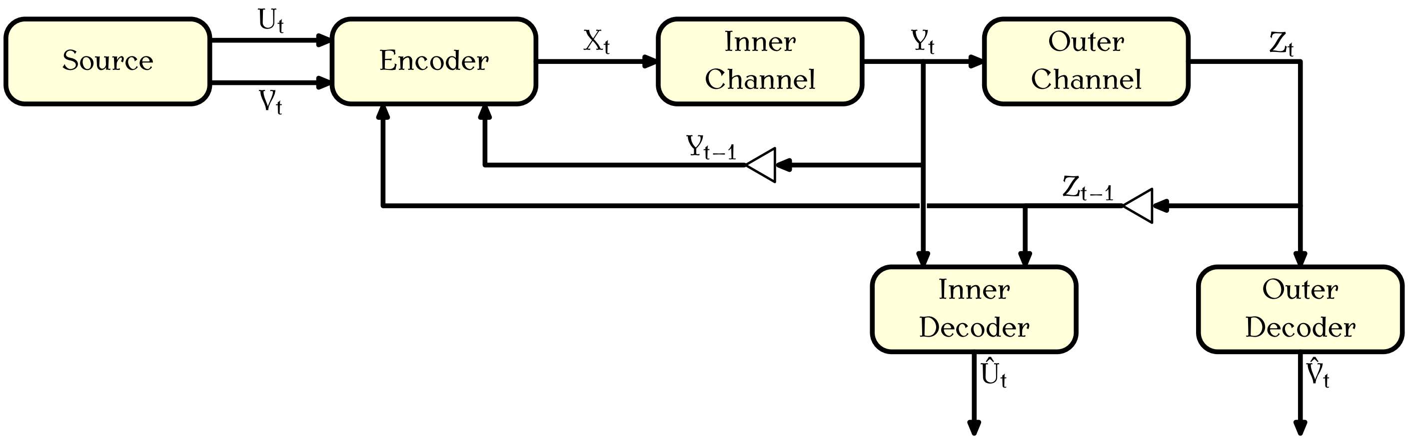
Can remove $(U^{t-1}, V^{t-1}, X^{t-1})$ without loss of optimality

Encoder: $X_t = c_t(U_t, V_t, Y^{t-1}, Z^{t-1})$

Decoders operate as before

Inner Dec: $\hat{U}_t = g_{1,t}(Y^t, Z^{t-1})$ Outer Dec: $\hat{V}_t = g_{2,t}(Z^t)$

Common observations

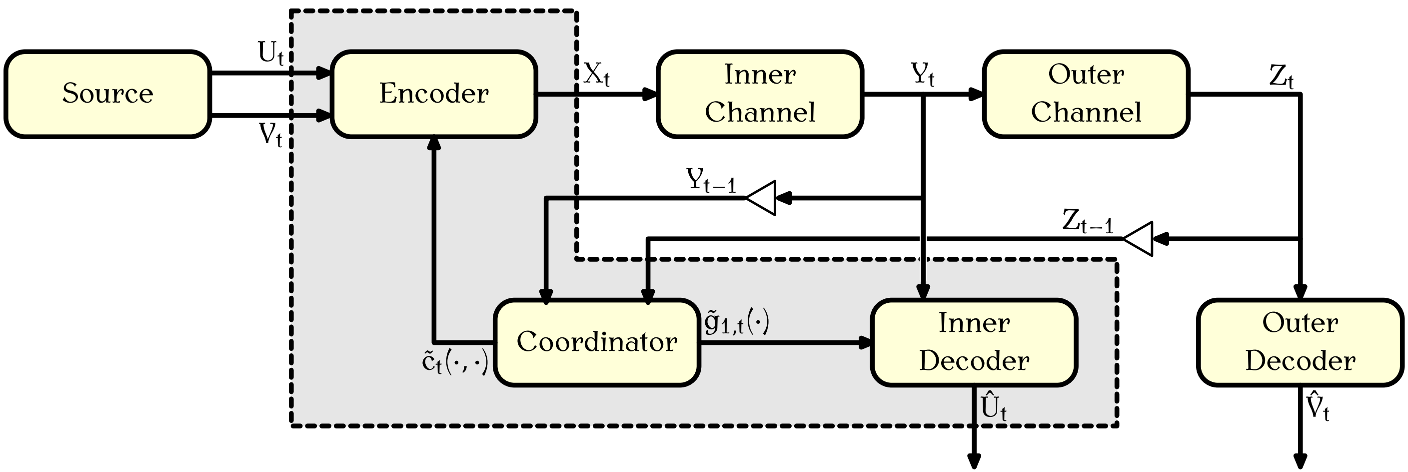


Encoder's observation: $(U_t, V_t, Y^{t-1}, Z^{t-1})$.

Inner encoder's observation: (Y^t, Z^{t-1}).

Common observation: (Y^{t-1}, Z^{t-1}).

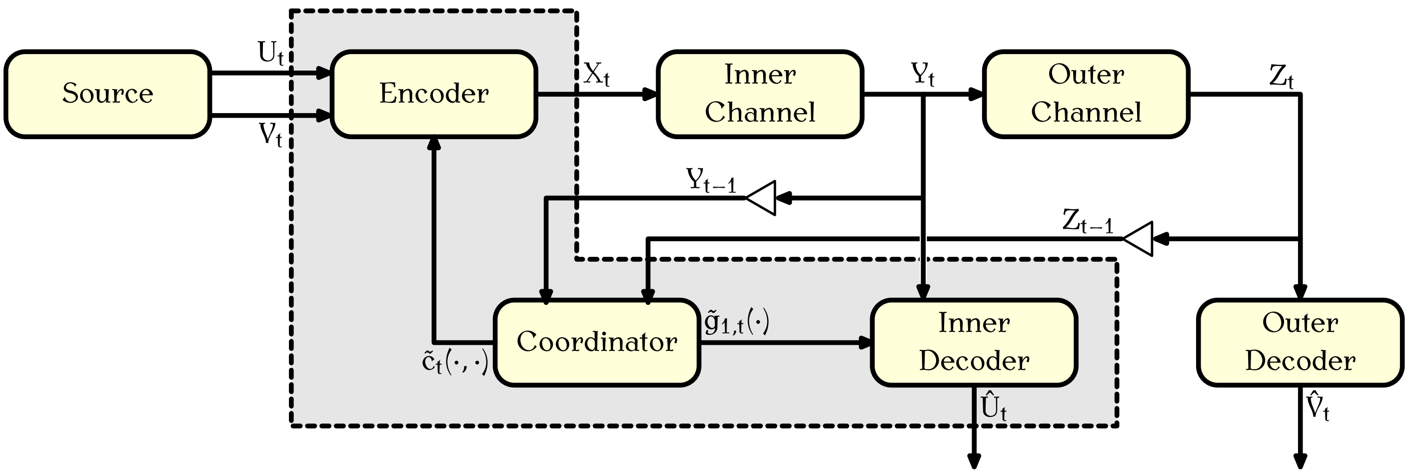
Step 2: A coordinator for enc and inner dec



Coordinator's observations: $(Y^{t-1},\ Z^{t-1})$

Coordinator's decisions: $\tilde{c}_t : \mathcal{U} \times \mathcal{V} \to \mathcal{X}$ and $\tilde{g}_{1,t} : \mathcal{Y} \to \hat{\mathcal{U}}.$

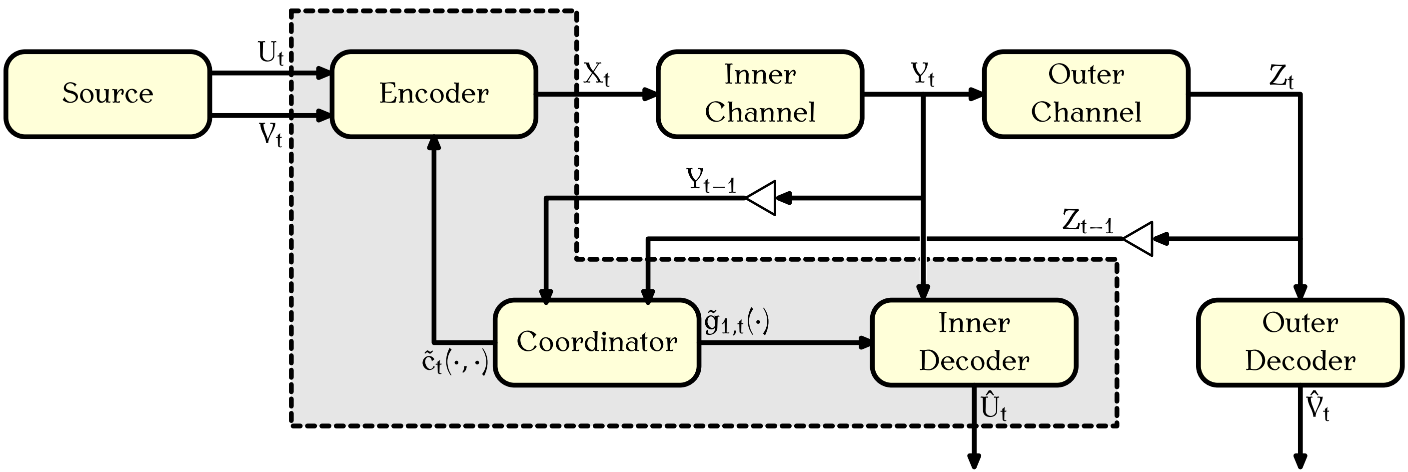
Step 2: A coordinator for enc and inner dec



Coordinator's information state: $\Xi_t(Y^t,\ Z^t;\ \tilde{c}^t) = \Pr(U_t,\ V_t \mid Y^t,\ Z^t;\ \tilde{c}^t)$

State Update: $\xi_t(y^t,\ z^t;\ \tilde{c}^t) = f_1\left(\ \xi_{t-1}(y^{t-1},\ z^{t-1};\ \tilde{c}^{t-1}),\ y_t,\ z_t,\ \tilde{c}_t\ \right).$

Step 2: A coordinator for enc and inner dec



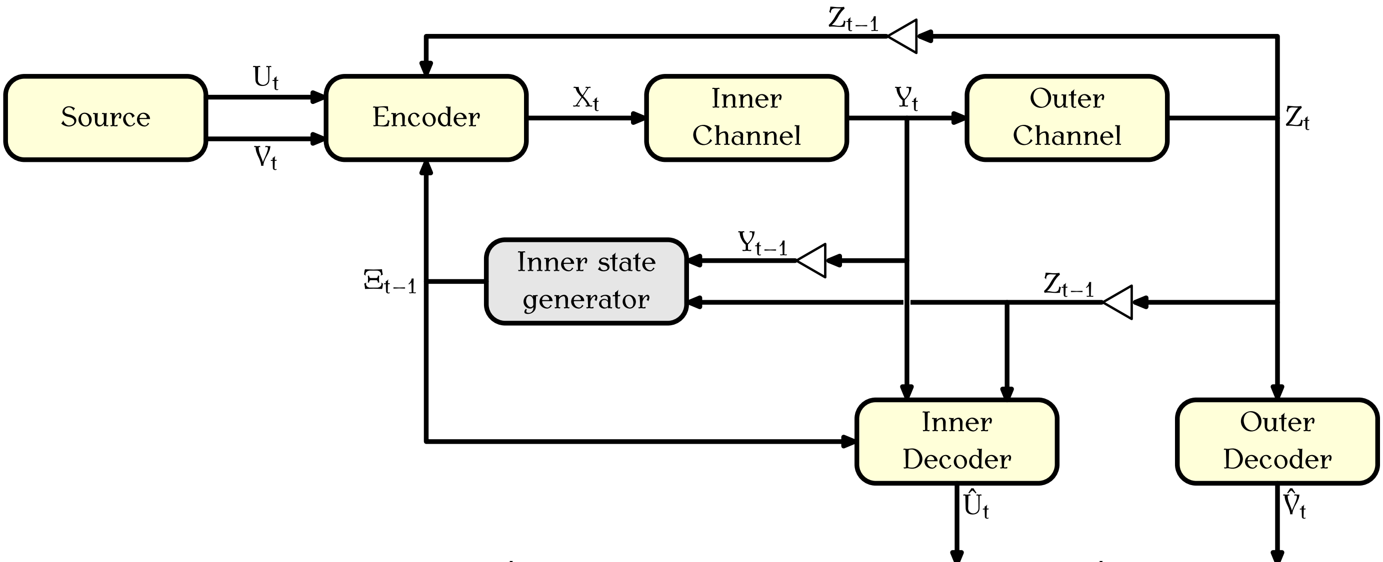
Outer decoder stay same as before $\quad \hat{V}_t = g_{2,t}(Z^t)$

Coordinator can replace Y^{t-1} with Ξ_{t-1} (but not Z^{t-1})

Coordinator's decisions:

$$(\tilde{c}_t,\ \tilde{g}_{1,t}) = \tilde{\phi}_t(\Xi_{t-1},\ Z^{t-1})$$

Back to the original system

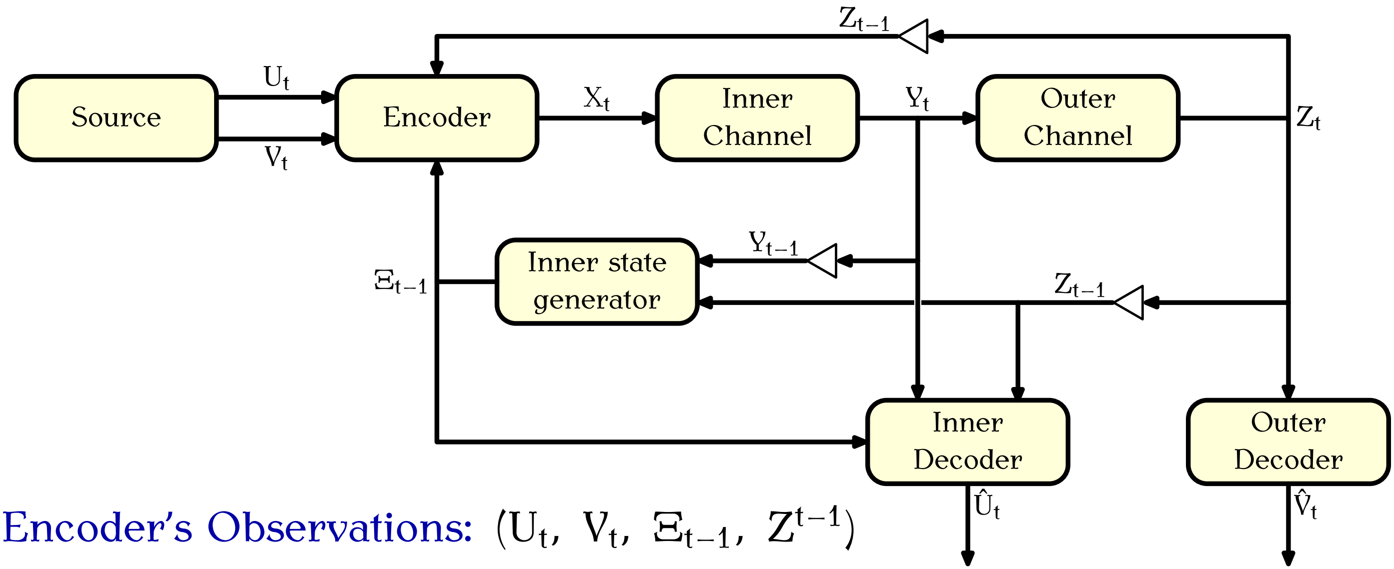


Encoder: $X_t = \tilde{c}_t(\Xi_{t-1}, Z^{t-1})(U_t, V_t) = c_t(U_t, V_t, \Xi_{t-1}, Z^{t-1})$

Inner decoder: $\hat{U}_t = \tilde{g}_{1,t}(\Xi_{t-1}, Z^{t-1})(Y_t) = g_{1,t}(Y_t, \Xi_{t-1}, Z^{t-1})$

Outer Decoder: $\hat{V}_t = g_{2,t}(Z^t)$

Common observations



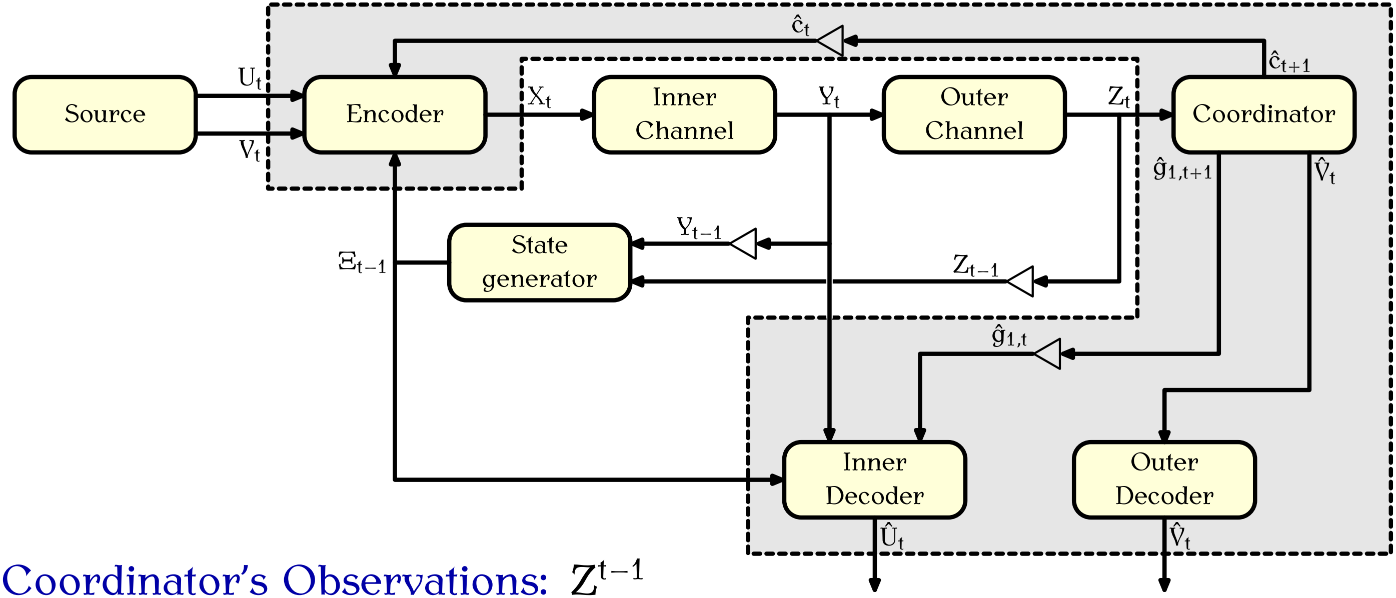
Encoder's Observations: $(U_t,\ V_t,\ \Xi_{t-1},\ Z^{t-1})$

Inner Decoder's Observations: $(Y_t,\ \Xi_{t-1},\ Z^{t-1})$

Outer Decoder's Observations: Z^t

Common Observations: Z^{t-1}

Step 3: A coordinator for enc and both decs



Coordinator's Observations: Z^{t-1}

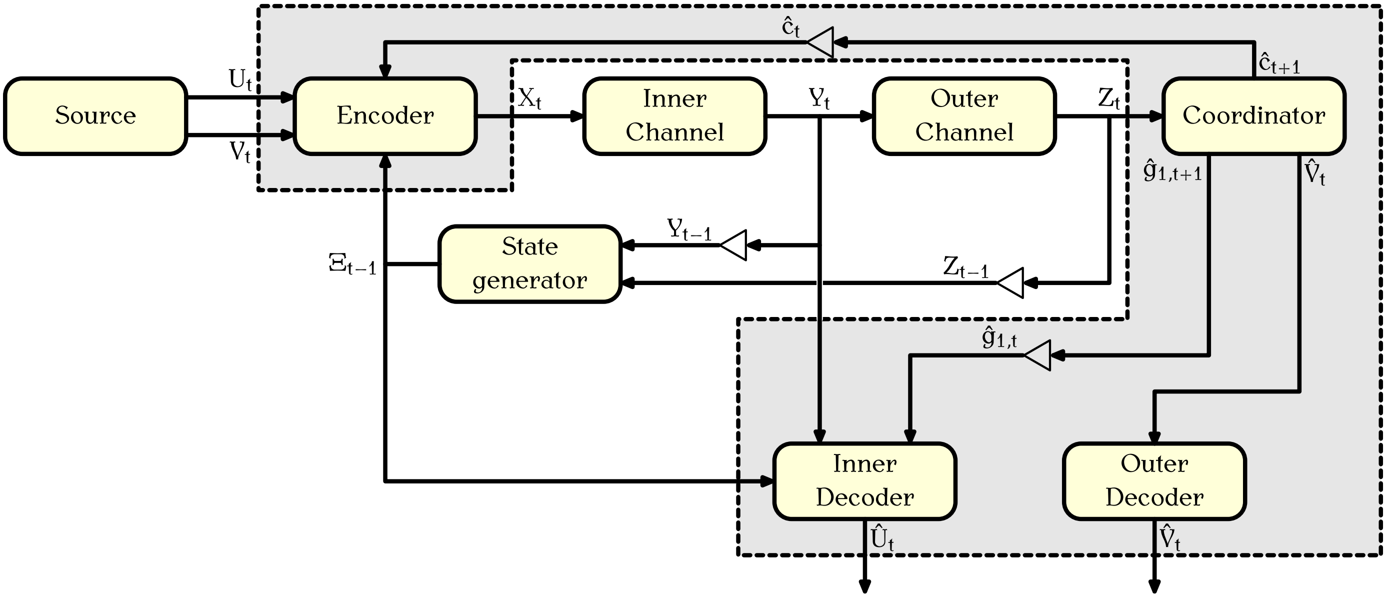
Coordinator's Decisions:

$\hat{c}_t : \mathcal{U} \times \mathcal{V} \times \Delta(\mathcal{U} \times \mathcal{V}) \to \mathcal{X}$

$\hat{g}_{1,t} : \mathcal{Y} \times \Delta(\mathcal{U} \times \mathcal{V}) \to \hat{\mathcal{U}}$

$\hat{V}_{t-1}$

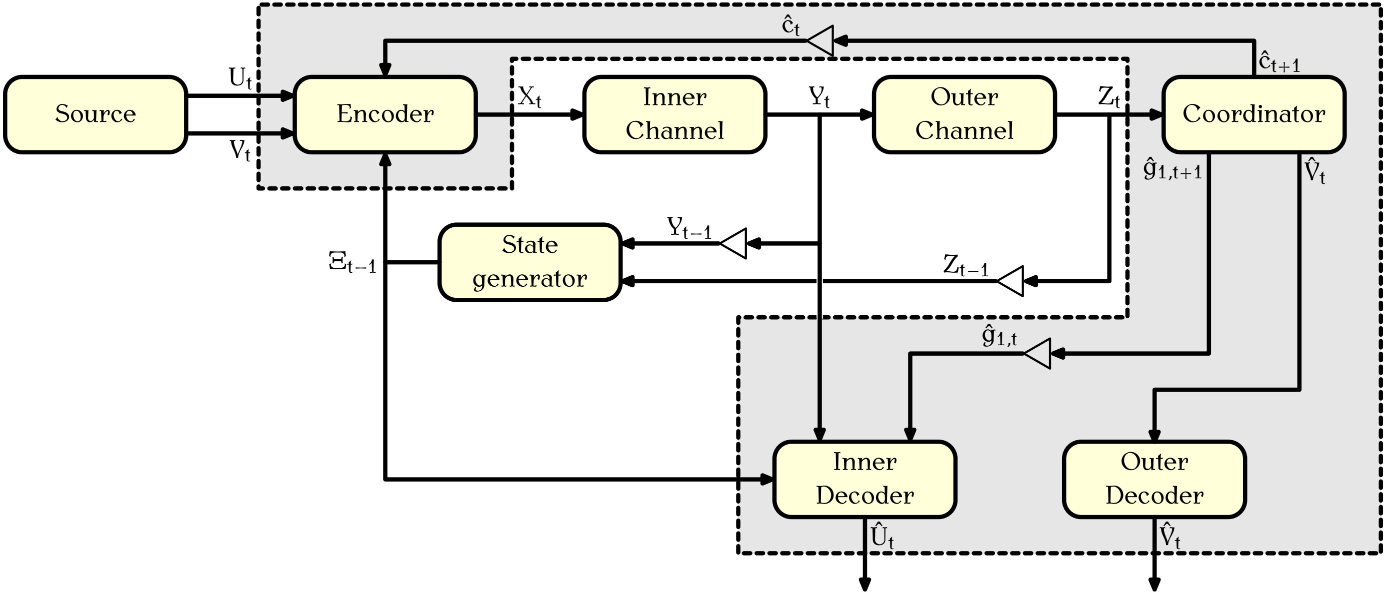
Step 3: A coordinator for enc and both decs



Info state: $\Pi_t(Z^t;\ \hat{c}^t) = \Pr(U_t,\ V_t,\ \Xi_t | Z^t;\ \hat{c}^t)$

State Update: $\pi_t(z^t;\ \hat{c}^t) = f_2\left(\ \pi_{t-1}(z^{t-1};\ \hat{c}^{t-1}),\ z_t,\ \hat{c}_t\ \right).$

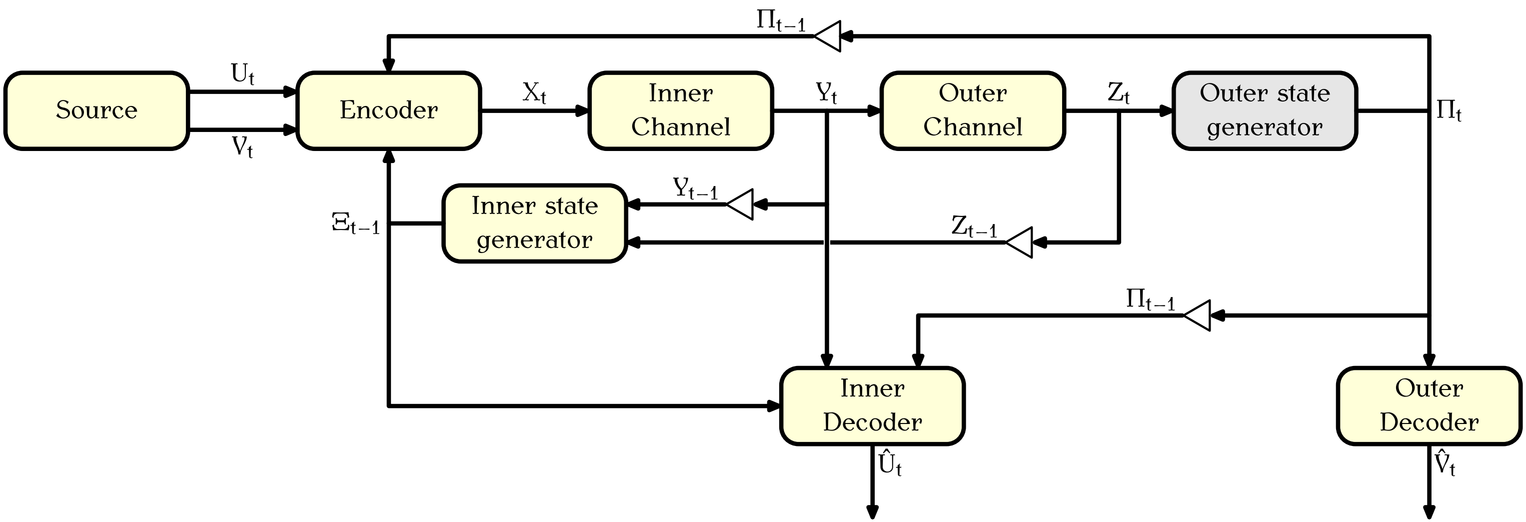
Step 3: A coordinator for enc and both decs



Coordinator's decisions:

$$(\hat{V}_{t-1},\ \hat{c}_t,\ \hat{g}_{1,t}) = \hat{\phi}(\Pi_{t-1})$$

Back to original system



Encoder: $X_t = \hat{c}_t(\Pi_{t-1})(U_t,\ V_t,\ \Xi_{t-1}) = c_t(U_t,\ V_t,\ \Xi_{t-1},\ \Pi_{t-1})$

Inner Decoder: $\hat{U}_t = \hat{g}_{1,t}(\Pi_{t-1})(Y_t,\ \Xi_{t-1}) = g_{1,t}(Y_t,\ \Xi_{t-1},\ \Pi_{t-1})$

Outer Decoder: $\hat{V}_t = g_{2,t}(\Pi_t)$

Step 4: Decoding is a filtration

Define:

$$\Theta_{1,t}(Y^t,\ Z^t;\ c^t) = \Pr(U_t \mid Y^t,\ Z^{t-1};\ c^t)$$

$$\Theta_{2,t}(Z^t;\ c^t) = \Pr(V_t \mid Z^t;\ c^t)$$

State Computation:

$$\theta_{1,t}(y^t,\ z^t;\ c^t) = h_1\left(\xi_{t-1}(y^{t-1},\ z^{t-1};\ \hat{c}^{t-1}),\ y_t,\ \hat{c}_t\right)$$

$$\theta_{2,t}(z^t;\ c^t) = h_2\left(\pi_t(z^t;\ \hat{c}^t)\right)$$

Optimal Decoding

$$\hat{U}_t = \tau_{1,t}(\Theta_{1,t})$$

$$\hat{V}_t = \tau_{2,t}(\Theta_{2,t})$$

where

$$\tau_{1,t}(\theta_1) = \arg\min_{\hat{u}\in\hat{\mathcal{U}}} \sum_{u\in\mathcal{U}} \rho_{1,t}(u,\hat{u})\theta_1(u)$$

$$\tau_{2,t}(\theta_2) = \arg\min_{\hat{v}\in\hat{\mathcal{V}}} \sum_{v\in\mathcal{V}} \rho_{2,t}(v,\hat{v})\theta_2(v)$$

Final structural result

Optimal strategy:

$$X_t = c_t(U_t,\ V_t,\ \Xi_{t-1},\ \Pi_{t-1})$$

$$\hat{U}_t = \tau_{1,t}(\Theta_{1,t})$$

$$\hat{V}_t = \tau_{2,t}(\Theta_{2,t})$$

State Computations:

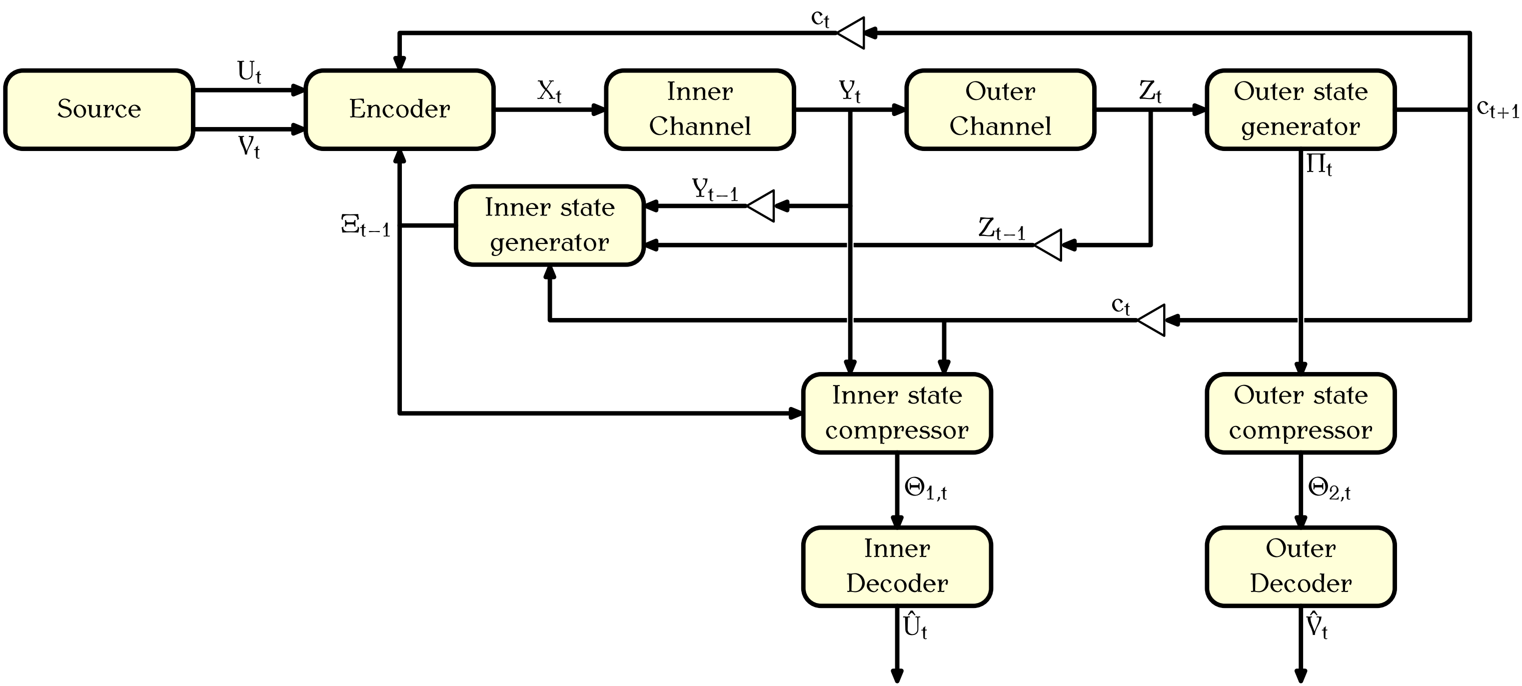
$$\xi_t = f_1(\xi_{t-1},\ y_t,\ z_t,\ \tilde{c}_t).$$

$$\pi_t = f_2(\pi_{t-1},\ z_t,\ \hat{c}_t).$$

$$\theta_{1,t} = h_1(\xi_{t-1},\ y_t,\ \hat{c}_t)$$

$$\theta_{2,t} = h_2(\pi_t)$$

Final structural result



Nodes of the system as a state machine

Node	State	Input	Control	Output
Source	(U_{t-1}, V_{t-1})	indep. noise	–	(U_t, V_t)
Encoder	–	(U_t, V_t, Ξ_{t-1})	$\hat{c}_t$	$X_t = \hat{c}_t(U_t, V_t, \Xi_{t-1})$
Inner Channel	–	$(X_t, N_{1,t})$	–	$Y_t = q_{1,t}(X_t, N_{1,t})$
Outer Channel	–	$(Y_t, N_{2,t})$	–	$Z_t = q_{2,t}(Y_t, N_{2,t})$
Inner state-gen.	Ξ_{t-1}	(Y_t, Z_t)	$\hat{c}_t$	$\Xi_t = f_1(\Xi_{t-1}, Y_t, Z_t, \hat{c}_t)$
Outer state-gen.	Π_{t-1}	Z_t	$\hat{c}_t$	$\Pi_t = f_2(\Pi_{t-1}, Z_t, \hat{c}_t)$
Inner state-compr.	–	(Y_t, Ξ_{t-1})	$\hat{c}_t$	$\Theta_{1,t} = h_1(\Xi_{t-1}, Y_t, \hat{c}_t)$
Outer state-compr.	–	Π_t	–	$\Theta_{2,t} = h_2(\Pi_t)$
Inner decoder	–	$\Theta_{1,t}$	–	$\hat{U}_t = \tau_{1,t}(\Theta_{1,t})$
Outer decoder	–	$\Theta_{2,t}$	–	$\hat{V}_t = \tau_{2,t}(\Theta_{2,t})$

Keep track of $(U_{t-1},\ V_{t-1},\ \Xi_{t-1},\ \Pi_{t-1})$ and $\hat{c}_t$

Sequential Decomposition

Π_{t-1} is sufficient to keep track of $(U_{t-1},\ V_{t-1},\ \Xi_{t-1},\ \Pi_{t-1})$ and $\hat{c}_t$

For $\pi \in \Delta(\mathcal{U} \times \mathcal{V} \times \Delta(\mathcal{U} \times \mathcal{V}))$ define

$$J_T(\pi) = \inf_{\hat{c}_T \in \hat{\mathcal{C}}} \hat{\rho}_T(\pi, \hat{c}_T),$$

and for $t = T-1, \ldots, 1$,

$$J_t(\pi) = \inf_{\hat{c}_t \in \hat{\mathcal{C}}} \left\{ \hat{\rho}_t(\pi, \hat{c}_t) + \sum_{\tilde{z} \in \mathcal{Z}} \hat{Q}(\tilde{z} | \pi; \hat{c}_t)\ J_{t+1}\left(f_2(\pi, \tilde{z}, \hat{c}_t) \right) \right\}$$

Conclusion

Presented structural result and sequential decomposition for broadcast channels with nested feedback.

May be useful in obtaining simple coding schemes for broadcast with feedback.

Motivates the study of general decentralized team problems with hierarchically nested information structures.