On real-time communication systems with noisy feedback

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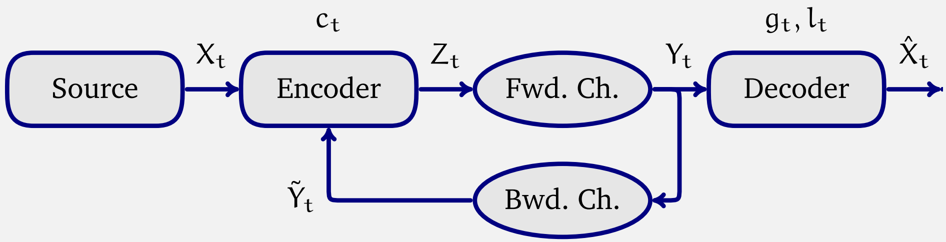
Problem Formulation



Model

Discrete time, discrete valued

Source	: $\{X_t, t = 1, \dots, T\}$
Encoder	: $Z_t = c_t(X^t, \tilde{Y}^{t-1}, Z^{t-1})$
Forward ch.	: $Y_t = h(Z_t, N_t)$
Backward ch.	: $\tilde{Y}_t = \tilde{h}(Y_t, \tilde{N}_t)$
Decoder	: $\hat{X}_t = g_t(Y_t, M_{t-1})$
Memory Up.	: $M_t = l_t(Y_t, M_{t-1})$

Objective

Choose $(c_1, \dots, c_T)$, $(g_1, \dots, g_T)$, $(l_1, \dots, l_T)$ to minimize

$$\mathcal{J} := \mathbf{E}\left\{\sum_{t=1}^{T} \rho(X_t, \hat{X}_t)\right\}$$

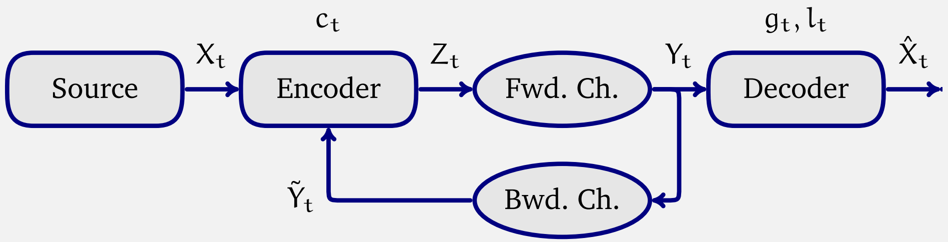
Outline

- Literature Overview
- Salient Features
- Solution Methodology

  (i) Structural Properties

  (ii) Global Optimization
- Conclusions

Literature Overview



real-time communication with **noisy feedback**

real-time communication

- *Types of problem*
  - (i) Source coding
  - (ii) Joint source-channel coding
- *Literature classification*
  - (i) Performance bounds
  - (ii) coding of individual sequences
  - (iii) coding of Markov sources

noisy feedback

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  - (i) Capacity of channels with memory and noisy feedback
  - (ii) Error exponents of memoryless channels with noisy feedback

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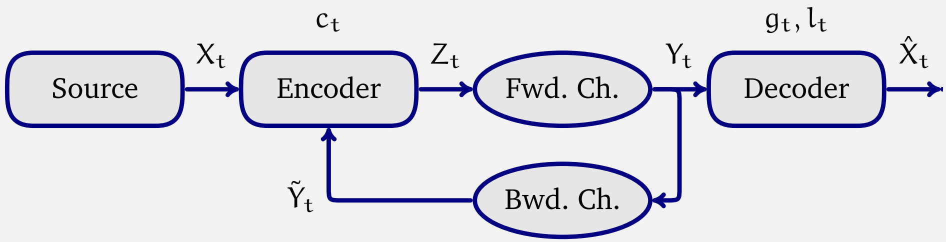

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- Real-time Constraint
- Finite Memory
- Noisy Feedback

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Do not know
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Information Theory

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- *Use Stochastic Optimization*

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- *Use Stochastic Optimization*

  Which solution methodology?

  - Markov Decision Theory
  - Orthogonal Search
  - Standard Form
  - ??

Stochastic Optimization

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  - ▷ works only for one decision maker with perfect recall

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- *Our Methodology*

  (i) Identify structural properties

  (ii) Use structural properties to solve the global optimization problem

Our Methodology
(i) Structural Properties

Structural Properties

Need

- Encoder: $Z_t = c_t(X^t, Z^{t-1}, \tilde{Y}^{t-1}), \qquad c_t \in \mathcal{C}_t : \mathcal{X}^t \times \mathcal{Z}^{t-1} \times \tilde{\mathcal{Y}}^{t-1} \to \mathcal{Z}$
- Domain changing with time.
- *Makes the infinite horizon design hard*
- *Can we simplify implementation?*

Preliminaries

- Notion of Information
- Notion of Beliefs

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- Let $(X_1, \ldots, X_T, N_1, \ldots, N_T, \tilde{N}_1, \ldots, \tilde{N}_T)$ be defined on $(\Omega, \mathfrak{F}, P)$.

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Information at Encoder

$$^1E_t := (X^t, Z^{t-1}, \tilde{Y}^{t-1}), \qquad ^1\mathfrak{E}_t := \sigma(^1E_t; {}^1\phi^{t-1}),$$
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$$^1R_t := (M_{t-1}), \qquad ^1\mathfrak{R}_t := \sigma(^1R_t; {}^1\phi^{t-1}),$$
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Beliefs — What one decision maker thinks about the data at other nodes

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Beliefs — What one decision maker thinks about the data at other nodes

Belief of the encoder

$$^iB_t(^ir) := \Pr\left(^iR_t = {}^ir \,\middle|\, {}^i\mathfrak{E}_t\right).$$

Belief of the decoder

$$^iA_t(^ie) := \Pr\left(^iE_t = {}^ie \,\middle|\, {}^i\mathfrak{R}_t\right),$$

Structure of Opt Encoders

- *The main idea*
  - Fix the decoding and memory update function
  - Look at the problem from the encoder's point of view
  - Derive qualitative properties of optimal encoders

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Structure of Optimal Encoders	$Z_t = c_t(X_t, {}^1B_t), \quad t = 2, \dots, T$

- We recover the structural results of previous models considered in literature
- *Systems where the encoder knows the decoder's information*
  - Real-time source coding (Witsenhausen, 1979)
  - Real-time joint source-channel coding with noiseless feedback (Walrand and Varaiya, 1982)
- *Systems where the encoder does not know the decoder's information*
  - Real-time joint source-channel coding with no feedback (Teneketzis, 2006)

Our Methodology

(ii) Global Optimization

Global Optimization

Information at Encoder

$$^{i}\mathfrak{E}_t = \sigma(^{i}E_t; {}^{i}\phi^{t-1})$$

Information at Decoder

$$^{i}\mathfrak{R}_t = \sigma(^{i}R_t; {}^{i}\phi^{t-1})$$

Global Optimization

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- Information is non nested. ${}^{i}\mathfrak{E}_t \not\subseteq {}^{i}\mathfrak{R}_t \quad {}^{i}\mathfrak{E}_t \not\supseteq {}^{i}\mathfrak{R}_t$

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Aumann's notion on Common Knowledge

$${}^{i}\mathfrak{K}_t \coloneqq {}^{i}\mathfrak{E}_t \cap {}^{i}\mathfrak{R}_t$$

- Choose future decision rules based on common knowledge.
- Or, since we are only interested in performance, choose future decision rules based on common belief: $P\big|_{{}^{i}\mathfrak{K}_t}$

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- Choose future decision rules based on common knowledge.
- Or, since we are only interested in performance, choose future decision rules based on common belief: $P\Big|_{{}^i\mathfrak{K}_t}$

- Need to find the set of all feasible realizations of common information (or common belief)
- ${}^i\mathfrak{E}_t$ and ${}^i\mathfrak{R}_t$ depend on past decision rules — so does ${}^i\mathfrak{K}_t$.

Global Optimization

Aumann's notion on Common Knowledge	${}^{i}\mathfrak{K}_t := {}^{i}\mathfrak{E}_t \cap {}^{i}\mathfrak{R}_t$

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- Choose Total Information: $^{i}\mathfrak{T}_t := \sigma(X_t, {}^{i}B_t, {}^{i}R_t; {}^{i}\phi_t) \supseteq {}^{i}\mathfrak{K}_t$
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The Image Space

- $(X_t, {}^{i}B_t, {}^{i}R_t) : (\Omega, \mathfrak{F}, P) \to (\mathcal{X} \times {}^{i}\mathcal{B} \times {}^{i}\mathcal{R},\ \mathbb{B}\left(\mathcal{X} \times {}^{i}\mathcal{B} \times {}^{i}\mathcal{R}\right),\ P')$

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- Only the measure P' depends on ${}^i\phi^{t-1}$
- Hard to determine set of feasible realizations.

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- Only the measure P' depends on ${}^{i}\phi^{t-1}$
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- Work with the set of all probability measures on

$$(\mathcal{X} \times {}^{i}\mathcal{B} \times {}^{i}\mathcal{R},\ \mathbb{B}\left(\mathcal{X} \times {}^{i}\mathcal{B} \times {}^{i}\mathcal{R}\right))$$

Global Optimization

Information States

$$^{1}\pi_t = \Pr\left(X_t, M_{t-1}, {}^{1}B_t\right),$$

$$^{2}\pi_t = \Pr\left(X_t, Y_t, M_{t-1}, {}^{2}B_t\right),$$

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$\Longrightarrow$

A Dynamic System

$$^{2}\pi_t = {}^{1}Q(c_t)\, {}^{1}\pi_t,$$
$$^{3}\pi_t = {}^{2}Q\, {}^{2}\pi_t,$$
$$^{1}\pi_{t+1} = {}^{3}Q(l_t)\, {}^{3}\pi_t.$$

$$\mathbf{E}\left\{\rho(X_t, \hat{X}_t) \,\middle|\, c^t, g^t, l^{t-1}\right\} = {}^{2}\rho({}^{2}\pi_t, g_t)$$

Global Optimization

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$$\mathbf{E}\left\{\rho(X_t, \hat{X}_t)\,\middle|\, c^t, g^t, l^{t-1}\right\} = {}^2\rho({}^2\pi_t, g_t)$$

An equivalent problem

Determine *meta-functions* $^1\Delta_t$, $^2\Delta_t$, and $^3\Delta_t$ such that:

$$c_t = {}^1\Delta_t({}^1\pi_t), \qquad {}^2\pi_t = {}^1Q(c_t)\,{}^1\pi_t,$$
$$g_t = {}^2\Delta_t({}^2\pi_t), \qquad {}^3\pi_t = {}^2Q\,{}^2\pi_t,$$
$$l_t = {}^3\Delta_t({}^3\pi_t), \qquad {}^1\pi_{t+1} = {}^3Q(l_t)\,{}^3\pi_t.$$

to minimize a total cost

$$\mathcal{J}_T(\Delta^T, {}^1\pi_1) = \sum_{t=1}^{T} {}^2\rho({}^2\pi_t, g_t).$$

Sequential Decomposition

$$
\begin{aligned}
{}^1V_t({}^1\pi) &= \inf_{c \in \mathcal{C}} {}^2V_t\left({}^1Q(c)\,{}^1\pi\right), \\
{}^2V_t({}^2\pi) &= \min_{g \in \mathcal{G}} {}^2\rho({}^2\pi, g) + {}^3V_t({}^2Q\,{}^2\pi), \\
{}^3V_t({}^3\pi) &= \min_{l \in \mathcal{L}} {}^1V_{t+1}\left({}^3Q(l)\,{}^3\pi\right).
\end{aligned}
$$

The arg min at each step determines an optimal meta-function

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- *Complexity*
  - Search complexity is linear in time (versus exponential in time for a brute force search)
  - Search complexity is exponential in the size of the alphabets

Extensions

- Infinite horizon
  - Expected discounted distortion
  - Average distortion per unit time
- Active feedback
- Channels with memory
- k-th order Markov source
- distortion with d-step delay

Conclusion

- An alternative approach to real-time communication
- Use stochastic optimization
  - (i) Derive structural results
  - (ii) Use structural results for global optimization
- A systematic search algorithm to determine optimal design

Future Directions

- Performance bounds
- Numerical algorithms
- Multi-terminal systems

Thank You