A Decision Theoretic Framework for Real-Time Communication

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Overview

What is Real-Time Communication?

- Real-Time (zero or finite delay) encoding,
- Real-Time (zero or finite-delay) decoding.

Why consider Real-Time Communication?

Motivated by informationally decentralized system

- QoS (delay) requirements in communication networks,
- Sensor networks,
- Traffic flow control in transportation networks,
- Decentralized resource allocation (decentralized routing)

Literature Overview

- Problem has received considerable attention in past.
    - Zero-delay and finite-delay source coding.
    - Causal Source coding.
    - Performance bounds of systems with a real-time or finite-delay constraint.
    - Zero-delay joint source channel coding.
    - Real-time quantization of Markov sources (noiseless channel).
    - Real-time encoding/decoding for noisy channels with noiseless feedback.
    - Real-time encoding/decoding for noisy channels (no feedback)

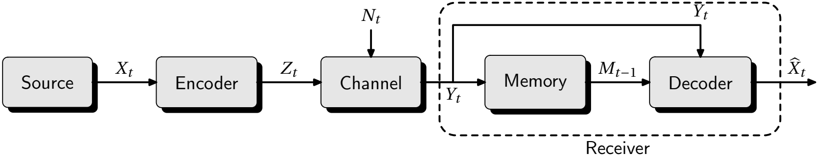
Literature Overview

- Problem has received considerable attention in past.
- Different approaches can be classified into two categories
  - Information Theoretic approach.
  - Decision Theoretic approach

Limitations of Standard Results of Information Theory

- Fundamental Results of Information Theory are asymptotic.
- Based on encoding/decoding of typical sequences.
- Small delay schemes work only when the time horizon goes to infinity.

System Model



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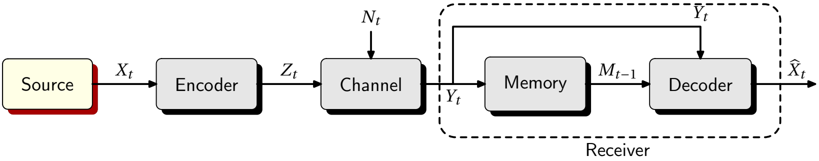
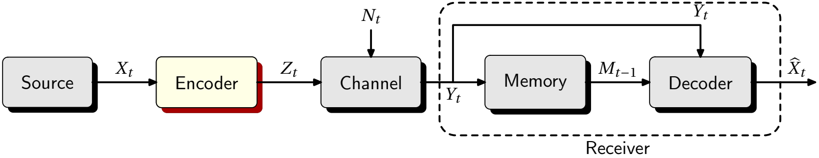


- Source is first order Markov with known statistics.

System Model



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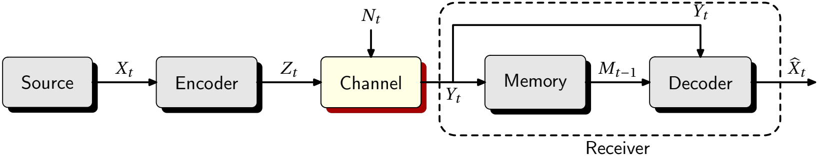
- Encoder is real-time

$$Z_t = c_t(X_1, X_2, \ldots, X_t)$$

System Model



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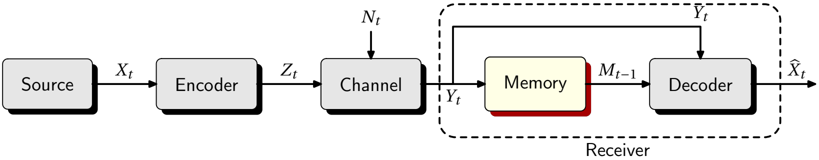
$$Z_t = c_t(X_1, X_2, \ldots, X_t)$$

- Discrete Memoryless Channel, known statistics.

$$\Pr\left(y_t \,\middle|\, x^t, z^t\right) = \Pr\left(y_t \,|\, z_t\right)$$

System Model



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- Finite memory receiver.

$$M_t = l_t(Y_t, M_{t-1})$$

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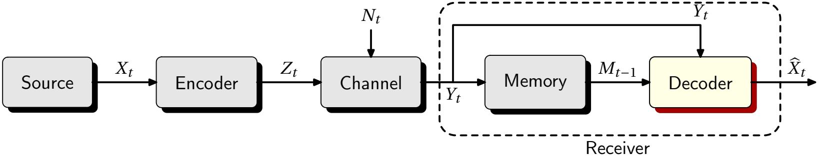

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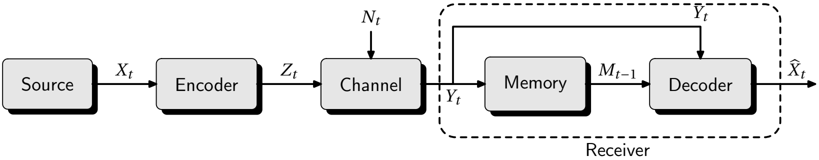
- Real-time decoder.

$$\hat{X}_t = g_t(Y_t, M_{t-1})$$

System Performance



- Distortion measure

$$\rho_t : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$$

- Design: Choice of $c \triangleq (c_1, c_2, \ldots, c_T)$, $g \triangleq (g_1, g_2, \ldots, g_T)$ and $l \triangleq (l_1, l_2, \ldots, l_T)$.
- Performance measure

$$\mathcal{J}(f, g, l) = \mathbb{E}\left\{ \sum_{t=1}^{T} \rho_t(X_t, \widehat{X}_t) \right\}$$

Problem Formulation

Problem

Assume that both encoder and decoder know

- statistics of the source,
- statistics of the channel,
- and the time horizon T

choose an optimal design (c^*, g^*, l^*) such that

$$\mathcal{J}^* = \mathcal{J}(c^*, g^*, l^*) = \min_{(c,g,l)} \mathcal{J}(c, g, l)$$

Salient Features

- dynamic team problem
- non-classical information structure
- non-convex (in policy space) optimization problem

Structural Results

D. Teneketzis.
On the structure of optimal real-time encoders and decoders in noisy communication.
submitted for publication in IEEE Trans. Inform. Theory.

Structure of Optimal Encoder

Consider any fixed (but arbitrarily) $g \triangleq (g_1, \ldots, g_T)$ and $l \triangleq (l_1, \ldots, l_T)$.

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Consider any fixed (but arbitrarily) $g \triangleq (g_1, \ldots, g_T)$ and $l \triangleq (l_1, \ldots, l_T)$. then,

There is no loss of optimality in restricting attention to encoding rules of the form

$$Z_t = c_t(X_t, P_{M_{t-1}}), \quad t = 2, 3, \ldots, T$$

where,

$$P_{M_t}(m) = \Pr\left(M_t = m \,\middle|\, X^t, Z^t, c^t, l^{t-1}\right)$$

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Structure of Optimal Decoder

Consider any fixed (but arbitrarily) $c \triangleq (c_1, \dots, c_T)$ and $l \triangleq (l_1, \dots, l_T)$, then

- Obtaining the optimal decoder is a filtering problem — At each t obtain g_t to minimize

$$\mathcal{J}_t = \mathbb{E}\left\{ \rho_t(X_t, \widehat{X}_t) \,\middle|\, Y_t = y, M_{t-1} = m \right\}$$

- An optimal decoding rule $g^* \triangleq (g_1^*, g_2^*, \dots, g_T^*)$ is given by

$$g_t^*(y_t, m_{t-1}) = \tau_t\big(\xi_t(y_t, m_{t-1})\big)$$

where $$\xi_t^{f,l}(y, m)(x) = \Pr\left(X_t = x \,|\, Y_t = y, m_{t-1} = m\right)$$

and $$\tau_t\big(\xi_t(y, m)\big) = \arg\min_a \sum_x \rho_t(x, a)\xi_t(y, m)(x)$$

Simplification of the Problem

Implication of Structural Results

- Without loss of optimality we can restrict attention to encoders of the form $Z_t = c_t(X_t, P_{M_{t-1}})$.
- Structure of optimal decoder depends only on the distortion measure and the conditional PMF ξ_t.
- ξ_t depends on choice of c^t, l^{t-1}.
- $g_t^* = g_t^*(c^t, l^{t-1})$
- $g^* = g^*(c, l)$

Equivalent Problem

$$\min_{(c,g,l)} \mathcal{J}(c, g, l) = \min_{c,l} \mathcal{J}\big(c, g^*(c, l), l\big)$$

Information States

Properties of Information States

- Need to obtain information states for both agents sufficient for performance evaluation.
- Let π_t and φ_t be information states of encoder and memory update respt. They should satisfy

  (S1a) π_t is a function of x^t, c^{t-1} and l^{t-1}.

  (S1b) φ_t is a function of y_t, m_{t-1}, c^t and l^{t-1}.

  (S2a) φ_t can be determined from π_t and c_t.

  (S2b) π_{t+1} can be determined from φ_t and l_t.

  (S3) $\cdots$

Information States

Properties of Information States

(S3) π_t absorbs the effect of c^{t-1}, l^{t-1} and φ_t absorbs the effect of c^t, l^{t-1} on expected future distortion, i.e.

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$$\begin{aligned}\mathbb{E}\left\{ \sum_{s=t}^{T} \rho_s(X_s, \widehat{X}_s) \,\middle|\, c, g, l \right\} &= \mathbb{E}\left\{ \sum_{s=t}^{T} \rho_s(X_s, \widehat{X}_s) \,\middle|\, \pi_t, c_t^T, l_t^T \right\} \\ &= \mathbb{E}\left\{ \sum_{s=t}^{T} \rho_s(X_s, \widehat{X}_s) \,\middle|\, \varphi_t, c_{t+1}^T, l_t^T \right\}\end{aligned}$$

Information States for the Problem

Consider the random vectors

$$P_{M_t}(m) = \Pr\left(M_t = m \,\middle|\, X^t, Z^t, c^t, l^{t-1}\right)$$
$$P_{Y_t, M_{t-1}}(y, m) = \Pr\left(Y_t = y, M_{t-1} = m \,\middle|\, X^t, Z^t, c^t, l^{t-1}\right)$$

Information States

$$\pi_t = \Pr\left(X_t, P_{M_{t-1}}\right), \qquad \text{(Info. state for Encoder)}$$
$$\varphi_t = \Pr\left(X_t, P_{Y_t, M_{t-1}}\right), \qquad \text{(Info. state for Memory Update)}$$

Information States for the Problem

π_t and φ_t satisfy (S1)–(S3), i.e.

1. there is a linear transformation $Q_t(c_t)$ such that

$$\varphi_t = Q_t(c_t)\pi_t$$

2. there is a linear transformation $\widehat{Q}_t(l_t)$ such that

$$\pi_{t+1} = \widehat{Q}_t(l_t)\varphi_t$$

3. for any choice of c and l, the expected conditional instantaneous cost can be expressed as

$$\mathbb{E}\left\{ \rho_t(X_t, \widehat{X}_t) \,\middle|\, c^t, g_t^*(c^t, l^{t-1}), l^{t-1} \right\} = \tilde{\rho}_t(\varphi_t)$$

where $g_t^*(c^t, l^{t-1})$ is an optimal decoding rule corresponding to c^t, l^{t-1} and $\tilde{\rho}_t(\cdot)$ is a deterministic function.

Equivalent Deterministic Problem

- System Equations

$$\varphi_t = Q_t(c_t)\pi_t, \qquad t = 1, \dots, T$$
$$\pi_{t+1} = \widehat{Q}_t(l_t)\varphi_t, \qquad t = 1, \dots, T-1$$

  $Q_t(\cdot)$ and $\widehat{Q}_t(\cdot)$ are deterministic transformations depending on c_t and l_t respt.
- Initial state π_1 is known.
- Instantaneous cost $\tilde{\rho}_t(\varphi_t)$.
- Optimization criterion

$$\inf_{c,l} \sum_{t=1}^{T} \tilde{\rho}_t(\varphi_t)$$

Nested Optimality Equations

$$
\begin{aligned}
\widehat{V}_T(\varphi) &\equiv 0 \\
V_t(\pi) &= \inf_{c_t} \Big[\tilde{\rho}_t\big(Q_t(c_t)\pi\big) + \widehat{V}_t\big(Q_t(c_t)\pi\big)\Big], \quad && t = 1, \dots, T \\
\widehat{V}_t(\varphi) &= \min_{l_t} \Big[V_{t+1}\big(\widehat{Q}_t(l_t)\varphi\big)\Big], && t = 1, \dots, T-1
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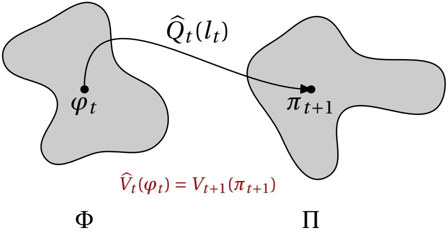

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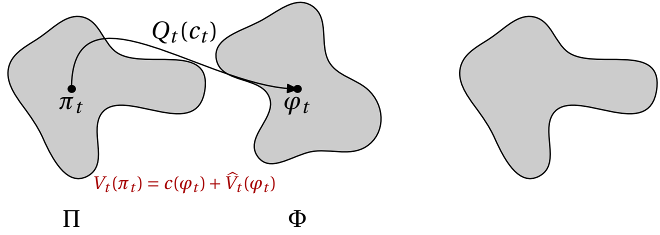

Time Homogenous Case

- source transition matrix
- channel
- noise statistics
- distortion measure

are time invariant. Then, the same methodology works for

- Finte time horizon,
- Infinite time horizon with an expected discounted distortion criterion.

Extensions

A. Mahajan and D. Teneketzis
On jointly optimal encoding, decoding and memory update for noisy real-time commuciation
Control Group Report CGR-05-07, Department of EECS, University of Michigan, Ann Arbor, MI.

- k-th order Markov source.
- Finite delay $\rho_t(X_{t-\delta}, \widehat{X}_t)$.
- Channels with memory.

Summary

- Provide a decision theoretic framework to study real-time communication.
- Use the stuctural results of Teneketzis 2004, to obtain jointly optimal real-time encoding, decoding and memory update rules.
- Extend the methodology to infinite horizon problems.

Future Work

- Extend the methodology to multi-terminal systems.
- Performance bounds.
- Computational issues.