Structure of optimal strategies for remote estimation over Gilbert-Elliott channel with feedback

Jhelum Chakravorty
Electrical and Computer Engineering
McGill University, Montreal, Canada
Email: jhelum.chakravorty@mail.mcgill.ca

Aditya Mahajan
Electrical and Computer Engineering
McGill University, Montreal, Canada
Email: aditya.mahajan@mcgill.ca

***Abstract*—We investigate remote estimation over a Gilbert-Elliot channel with feedback. The channel is modelled as an ON/OFF channel, where the state of the channel evolves as a Markov chain. The channel state is observed by the receiver and fed back to the transmitter with one unit delay. In addition, the transmitter gets ACK/NACK feedback for successful/unsuccessful transmission. Using ideas from team theory, we establish the structure of optimal transmission and estimation strategies and identify a dynamic program to determine optimal strategies with that structure. We then consider first-order autoregressive sources where the noise process has unimodal and symmetric distribution. Using ideas from majorization theory, we show that the optimal transmission strategy has a threshold structure and the optimal estimation strategy is Kalman-filter like.**

I. Introduction

A. Motivation and literature overview

We consider a remote estimation system in which a sensor/transmitter observes a first-order Markov process and causally decides which observations to transmit to a remotely located receiver/estimator. Communication is expensive and takes place over a Gilbert-Elliot channel (which is used to model channels with burst erasures). The channel has two states—OFF state and ON state—and the state evolves as a Markov chain. When the channel is in the OFF state, a packet transmitted from the sensor to the receiver is dropped. When the channel is in the ON state, a packet transmitted from the sensor to the receiver is received without error. We assume that the channel state is causally observed at the receiver and is fed back to the transmitter with one-unit delay. Whenever there is a successful reception, the receiver sends an acknowledgment to the transmitter. The feedback is assumed to be noiseless.

At the time instances when the receiver does not receive a packet (either because the sensor did not transmit or because the transmitted packet was dropped), it needs to estimate the state of the source process. There is a fundamental trade-off between communication cost and estimation accuracy. Transmitting all the time minimizes the estimation error but incurs a high communication cost; not transmitting at all minimizes the communication cost but incurs a high estimation error.

The motivation of remote estimation comes from networked control systems. The earliest instance of the problem was perhaps considered by Marschak [1]. In recent years, several variations of remote estimation has been considered. These include models that consider idealized channels without packet drops [2]–[9] (also see references therein), models that consider channels with packet drops [10]–[12] and models that consider channels with noise [13]–[15].

The salient features of remote estimation are as follows: (F1) The decisions are made sequentially. (F2) The reconstruction/estimation at the receiver must be done with zero-delay. (F3) When a packet does get through, it is received without noise (this feature is absent in channels with noise).

Remote estimation problems may be viewed as a special case of real-time communication [16]–[19] (and references therein). As in real-time communication, the key conceptual difficulty is that the data available at the transmitter and the receiver is increasing with time. Thus, the domain of the transmission and the estimation function increases with time.

To circumvent this difficulty one needs to identify sufficient statistics for the data at the transmitter and the data at the receiver. In the real-time communication literature, dynamic team theory (or decentralized stochastic control theory) is used to identify such sufficient statistics as well as to identify a dynamic program to determine the optimal transmission and estimation strategies. Similar ideas are also used in remote-estimation literature. In addition, feature (F3) allows one to further simplify the structure of optimal transmission and estimation strategies. When the source is a first-order autoregressive process, majorization theory is used to show that the optimal transmission strategies is characterized by a threshold [5]–[7], [9]–[11]. In particular, it is optimal to transmit when the instantaneous distortion due to not transmitting is greater than a threshold. The optimal thresholds can be computed either using dynamic programming [5], [6] or using renewal relationships [9], [10]. In this paper, we consider packet drop channels with Markovian memory. We identify sufficient statistics at the transmitter and the receiver. When the source is a first-order autoregressive process, we show that threshold-based strategies (where the threshold depends on the previous channel-state) are optimal.

A model very close to ours is considered in [11], which investigates remote estimation over a fading channel. The fading gains evolve as a Markov chain and the objective is to choose the transmission power (the binary transmission model that is considered here is then a special case). In [11], it is assumed that the transmitter knows the current realization of the channel state; in our model we assume that the transmitter knows the one-step delayed channel state information. In [6], an infinite horizon long-term average cost model is considered; we consider a finite horizon model.

Another closely related model is considered in [12], which investigates remote estimation of a hidden Markov state process over a Gilbert-Elliot channel. However, in [12] attention is restricted to a stochastic event triggered transmission strategy and the performance of such strategies is evaluated using a change of measure argument. In contrast, we do not assume a specific form of transmission strategy.

B. The communication system

1) Source model: The source is a first-order time-homogeneous Markov process $\{X_t\}_{t\geq 0}$, $X_t \in \mathcal{X}$. For ease of exposition, in the first part of the paper we assume that $\mathcal{X}$ is a finite set. We will later argue that a similar argument works when $\mathcal{X}$ is a general measurable space. The transition probability matrix of the source is denoted by P, i.e., for any $x, y \in \mathcal{X}$, $P_{xy} \coloneqq \mathbb{P}(X_{t+1} = y \mid X_t = x)$.

2) Channel model: The channel is a Gilbert-Elliott channel [20], [21]. The channel state $\{S_t\}_{t\geq 0}$ is a binary-valued time-homogeneous Markov process. We use the convention that $S_t = 0$ denotes that the channel is in the OFF state and $S_t = 1$ denotes that the channel is in the ON state. The transition probability matrix of the channel state is denoted by Q, i.e., for $r, s \in \{0, 1\}$, $Q_{rs} \coloneqq \mathbb{P}(S_{t+1} = s | S_t = r)$.

The input alphabet $\bar{\mathcal{X}}$ of the channel is $\mathcal{X} \cup \{\mathfrak{E}\}$, where $\mathfrak{E}$ denotes the event that there is no transmission. The channel output alphabet $\mathcal{Y}$ is $\mathcal{X} \cup \{\mathfrak{E}_0, \mathfrak{E}_1\}$, where the symbols $\mathfrak{E}_0$ and $\mathfrak{E}_1$ are explained below. At time t, the channel input is denoted by $\bar{X}_t$ and the channel output is denoted by Y_t.

The channel is a channel with state. In particular, for any realization $(\bar{x}_{0:T}, s_{0:T}, y_{0:T})$ of $(\bar{X}_{0:T}, S_{0:T}, Y_{0:T})$, we have

$$\begin{aligned}\mathbb{P}(Y_t = y_t \mid \bar{X}_{0:t} = \bar{x}_{0:t}, S_{0:t} = s_{0:t}) \\ = \mathbb{P}(Y_t = y_t \mid \bar{X}_t = \bar{x}_t, S_t = s_t) \quad (1)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(S_t = s_t \mid \bar{X}_{0:t} = \bar{x}_{0:t}, S_{0:t-1} = s_{0:t-1}) \\ = \mathbb{P}(S_t = s_t \mid S_{t-1} = s_{t-1}) = Q_{s_{t-1}s_t} \quad (2)\end{aligned}$$

Note that the channel output Y_t is a deterministic function of the input $\bar{X}_t$ and the state S_t. In particular, for any $\bar{x} \in \bar{\mathcal{X}}$ and $s \in \{0, 1\}$, the channel output y is given as follows:

$$y = \begin{cases} \bar{x}, & \text{if } \bar{x} \in \mathcal{X} \text{ and } s = 1 \\ \mathfrak{E}_1, & \text{if } \bar{x} = \mathfrak{E} \text{ and } s = 1 \\ \mathfrak{E}_0, & \text{if } s = 0 \end{cases}$$

This means that if there is a transmission (i.e., $\bar{x} \in \mathcal{X}$) and the channel is on (i.e., $s = 1$), then the receiver observes $\bar{x}$. However, if there is no transmission (i.e., $\bar{x} = \mathfrak{E}$) and the channel is on (i.e., $s = 1$), then the receiver observes $\mathfrak{E}_1$. If the channel is off, then the receiver observes $\mathfrak{E}_0$.

3) The transmitter: There is no need for channel coding in a remote-estimation setup. Instead, the role of the transmitter is to determine which source realizations need to be transmitted. Let $U_t \in \{0, 1\}$ denote the transmitter's decision. We use the convention that $U_t = 0$ denotes that there is no transmission (i.e., $\bar{X}_t = \mathfrak{E}$) and $U_1 = 1$ denotes that there is transmission (i.e., $\bar{X}_t = X_t$).

Transmission is costly. Each time the transmitter transmits (i.e., $U_t = 1$), it incurs a cost of λ.

4) The receiver: At time t, the receiver generates an estimate $\hat{X}_t \in \mathcal{X}$ of X_t. The quality of the estimate is determined by a distortion function $d \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$.

C. Information structure and problem formulation

It is assumed that the receiver observes the channel state causally. Thus, the information available at the receiver[1] is

$$I^2_t = \{S_{0:t}, Y_{0:t}\}.$$

The estimate $\hat{X}_t$ is chosen according to

$$\hat{X}_t = g_t(I^2_t) = g_t(S_{0:t}, Y_{0:t}), \tag{3}$$

where g_t is called the *estimation rule* at time t. The collection $\boldsymbol{g} \coloneqq (g_1, \dots, g_T)$ for all time is called the *estimation strategy*.

It is assumed that there is one-step delayed feedback from the receiver to the transmitter.[2] Thus, the information available at the transmitter is

$$I^1_t = \{X_{0:t}, U_{0:t-1}, S_{0:t-1}, Y_{0:t-1}\}.$$

The transmission decision U_t is chosen according to

$$U_t = f_t(I^1_t) = f_t(X_{0:t}, U_{0:t-1}, S_{0:t-1}, Y_{0:t-1}), \tag{4}$$

where f_t is called the *transmission rule* at time t. The collection $\boldsymbol{f} \coloneqq (f_1, \dots, f_T)$ for all time is called the *transmission strategy*.

The collection $(\boldsymbol{f}, \boldsymbol{g})$ is called a *communication strategy*. The performance of any communication strategy $(\boldsymbol{f}, \boldsymbol{g})$ is given by

$$J(\boldsymbol{f}, \boldsymbol{g}) = \mathbb{E}\Big[\sum_{t=0}^{T} \lambda U_t + d(X_t, \hat{X}_t)\Big] \tag{5}$$

where the expectation is taken with respect to the joint measure on all system variables induced by the choice of $(\boldsymbol{f}, \boldsymbol{g})$.

We are interested in the following optimization problem.

Problem 1 *In the model described above, identify a communication strategy $(\boldsymbol{f}^*, \boldsymbol{g}^*)$ that minimizes the cost $J(\boldsymbol{f}, \boldsymbol{g})$ defined in* (5).

[1] We use superscript 1 to denote variables at the transmitter and superscript 2 to denote variables at the receiver.

[2] Note that feedback requires two bits: the channel state S_t is binary and the channel output Y_t can be communicated by indicating whether $Y_t \in \mathcal{X}$ or not (i.e., transmitting an ACK or a NACK).

II. Main results

A. Structure of optimal communication strategies

Two-types of structural results are established in the real-time communication literature: (i) establishing that part of the data at the transmitter is irrelevant and can be dropped without any loss of optimality; (ii) establishing that the common information between the transmitter and the receiver can be "compressed" using a belief state. The first structural results were first established by Witsenhausen [16] while the second structural results were first established by Walrand Varaiya [17]. We establish both types of structural results for remote estimation.

Lemma 1 *For any estimation strategy of the form* (3)*, there is no loss of optimality in restricting attention to transmission strategies of the form*

$$U_t = f_t(X_t, S_{0:t-1}, Y_{0:t-1}). \tag{6}$$

The proof idea is similar to [18]. We show that $\{X_t, S_{0:t-1}, Y_{0:t-1}\}_{t\geq 0}$ is a controlled Markov process controlled by $\{U_t\}_{t\geq 0}$. See [22] for proof.

Now, following the common information approach of [23], for any transmission strategy $\boldsymbol{f}$ of the form (6) and any realization $(s_{0:T}, y_{0:T})$ of $(S_{0:T}, Y_{0:T})$, define $\varphi_t \colon \mathcal{X} \to \{0,1\}$ as

$$\varphi_t(x) = f_t(x, s_{0:t-1}, y_{0:t-1}), \quad \forall x \in \mathcal{X}.$$

Furthermore, define conditional probability measures π^1_t and π^2_t on $\mathcal{X}$ as follows: for any $x \in \mathcal{X}$,

$$\pi^1_t(x) \coloneqq \mathbb{P}^{\boldsymbol{f}}(X_t = x \mid S_{0:t-1} = s_{0:t-1}, Y_{0:t-1} = y_{0:t-1}),$$
$$\pi^2_t(x) \coloneqq \mathbb{P}^{\boldsymbol{f}}(X_t = x \mid S_{0:t} = s_{0:t}, Y_{0:t} = y_{0:t}).$$

We call π^1_t the *pre-transmission belief* and π^2 the *post-transmission belief*. Note that when $(S_{0:T}, Y_{0:T})$ are random variables, then π^1_t and π^2_t are also random variables which we denote by Π^1_t and Π^2_t.

For the ease of notation, for any $\varphi\colon \mathcal{X} \to \{0,1\}$ and $i \in \{0,1\}$, define the following:

- $B_i(\varphi) = \{x \in \mathcal{X} : \varphi(x) = i\}$.
- For any probability distribution π on $\mathcal{X}$ and any subset $\mathcal{A}$ of $\mathcal{X}$, $\pi(\mathcal{A})$ denotes $\sum_{x\in\mathcal{A}} \pi(x)$.
- For any probability distribution π on $\mathcal{X}$, $\xi = \pi|_\varphi$ means that $\xi(x) = \mathbb{1}_{\{\varphi(x)=0\}}\pi(x)/\pi(B_0(\varphi))$.

Lemma 2 *Given any transmission strategy $\boldsymbol{f}$ of the form* (6)*:*

1) *there exists a function F^1 such that*

$$\pi^1_{t+1} = F^1(\pi^2_t) = \pi^2_t P. \tag{7}$$

2) *there exists a function F^2 such that*

$$\pi^2_t = F^2(\pi^1_t, \varphi_t, y_t) = \begin{cases} \delta_{y_t} & \text{if } y_t \in \mathcal{X} \\ \pi^1_t|_{\varphi_t}, & \text{if } y_t = \mathfrak{E}_1 \\ \pi^1_t, & \text{if } y_t = \mathfrak{E}_0. \end{cases} \tag{8}$$

Note that in (7), we are treating π^2_t as a row-vector and in (8), δ_{y_t} denotes a Dirac measure centered at y_t. The update equations (7) and (8) are standard non-linear filtering equations. See [22] for proof.

Theorem 1 *In Problem 1, we have that:*

1) Structure of optimal strategies: *There is no loss of optimality in restricting attention to optimal transmission and estimation strategies of the form:*

$$U_t = f^*_t(X_t, S_{t-1}, \Pi^1_t), \tag{9}$$
$$\hat{X}_t = g^*_t(\Pi^2_t). \tag{10}$$

2) Dynamic program: *Let $\Delta(\mathcal{X})$ denote the space of probability distributions on $\mathcal{X}$. Define value functions $V^1_t \colon \{0,1\} \times \Delta(\mathcal{X}) \to \mathbb{R}$ and $V^2_t \colon \{0,1\} \times \Delta(\mathcal{X}) \to \mathbb{R}$ as follows.*

$$V^1_{T+1}(s, \pi^1) = 0, \tag{11}$$

and for $t \in \{T, \dots, 0\}$

$$V^1_t(s, \pi^1) = \min_{\varphi\colon \mathcal{X}\to\{0,1\}} \Big\{ \lambda \pi^1(B_1(\varphi)) + W^0_t(\pi^1, \varphi)\pi^1(B_0(\varphi)) + \sum_{x\in B_1(\varphi)} W^1_t(\pi^1, \varphi, x)\pi^1(x) \Big\} \tag{12}$$

$$V^2_t(s, \pi^2) = \min_{\hat{x}\in\mathcal{X}} \sum_{x\in\mathcal{X}} d(x, \hat{x})\pi^2(x) + V^1_{t+1}(s, \pi^2 P), \tag{13}$$

where,

$$W^0_t(\pi^1, \varphi) = Q_{s0}V^2_t(0, \pi^1) + Q_{s1}V^2_t(1, \pi^1|_\varphi),$$
$$W^1_t(\pi^1, \varphi, x) = Q_{s0}V^2_t(0, \pi^1) + Q_{s1}V^2_t(1, \delta_x).$$

Let $\Psi_t(s, \pi^1)$ denote the arg min of the right hand side of (12)*. Then, the optimal transmission strategy of the form* (9) *is given by*

$$f^*_t(\cdot, s, \pi^1) = \Psi_t(s, \pi^1).$$

Furthermore, the optimal estimation strategy of the form (10) *is given by*

$$g^*_t(\pi^2) = \arg\min_{\hat{x}\in\mathcal{X}} \sum_{x\in\mathcal{X}} d(x, \hat{x})\pi^2(x). \tag{14}$$

The proof idea is as follows. Once we restrict attention to transmission strategies of the form (6), the information structure is partial history sharing [23]. Thus, one can use the common information approach of [23] and obtain the structure of optimal strategies. See [22] for proof.

Remark 1 Although the above model and result are stated for sources with finite alphabets, they extend naturally to general state spaces (including Euclidean spaces) under standard technical assumptions. See [24] for details.

B. Optimality of threshold-based strategies for autoregressive source

In this section, we consider a first-order autoregressive source $\{X_t\}_{t\geq 0}$, $X_t \in \mathbb{R}$, where the initial state $X_0 = 0$ and for $t \geq 0$, we have that

$$X_{t+1} = aX_t + W_t, \tag{15}$$

where $a \in \mathbb{R}$ and $W_t \in \mathbb{R}$ is distributed according to a symmetric and unimodal distribution with probability density function μ. Furthermore, the per-step distortion is given by $d(X_t - \hat{X}_t)$, where $d(\cdot)$ is an even function that is increasing on $\mathbb{R}_{\geq 0}$. The rest of the model is the same as before.

For the above model, we can further simplify the result of Theorem 1, as given by Theorem 2. See Section III for the proof.

Theorem 2 *For a first-order autoregressive source with symmetric and unimodal disturbance,*

1) Structure of optimal estimation strategy: *The optimal estimation strategy is given as follows: $\hat{X}_0 = 0$, and for $t \geq 0$,*

$$\hat{X}_t = \begin{cases} a\hat{X}_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathbb{R} \end{cases} \tag{16}$$

2) Structure of optimal transmission strategy: *There exist threshold functions $k_t \colon \{0,1\} \to \mathbb{R}_{\geq 0}$ such that the following transmission strategy is optimal:*

$$f_t(X_t, S_{t-1}, \Pi_t^1) = \begin{cases} 1, & \text{if } |X_t - a\hat{X}_{t-1}| \geq k_t(S_{t-1}) \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Remark 2 It can be shown that under the optimal strategy, Π_t^2 is symmetric and unimodal (SU) (Definition 1) around $\hat{X}_t$ and, therefore, Π_t^1 is SU around $a\hat{X}_{t-1}$. Thus, the transmission and estimation strategies in Theorem 2 depend on the pre- and post-transmission beliefs only through their means.

III. Proof of Theorem 2

A. A change of variables

Define a process $\{Z_t\}_{t\geq 0}$ as follows: $Z_0 = 0$ and for $t \geq 0$,

$$Z_t = \begin{cases} aZ_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathcal{X} \end{cases}$$

Note that Z_t is a function of $Y_{0:t-1}$. Next, define processes $\{E_t\}_{t\geq 0}$, $\{E_t^+\}_{t\geq 0}$, and $\{\hat{E}_t\}_{t\geq 0}$ as follows:

$$E_t \coloneqq X_t - aZ_{t-1}, \quad E_t^+ \coloneqq X_t - Z_t, \quad \hat{E}_t \coloneqq \hat{X}_t - Z_t$$

The processes $\{E_t\}_{t\geq 0}$ and $\{E_t^+\}_{t\geq 0}$ are related as follows: $E_0 = 0$, $E_0^+ = 0$, and for $t \geq 0$

$$E_t^+ = \begin{cases} E_t, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ 0, & \text{if } Y_t \in \mathcal{X} \end{cases} \quad \text{and} \quad E_{t+1} = aE_t^+ + W_t.$$

Since $X_t - \hat{X}_t = E_t^+ - \hat{E}_t$, we have that $d(X_t - \hat{X}_t) = d(E_t^+ - \hat{E}_t)$.

Next, redefine the pre- and post-transmission beliefs in terms of the error process. With a slight abuse of notation, we still denote the probability density of the pre- and post-transmission beliefs as π_t^1 and π_t^2. In particular, π_t^1 is the conditional pdf of E_t given $(s_{0:t-1}, y_{0:t-1})$ and π_t^2 is the conditional pdf of E_t^+ given $(s_{0:t}, y_{0:t})$.

Let $H_t \in \{\mathfrak{E}_0, \mathfrak{E}_1, 1\}$ denote the event whether the transmission was successful or not. In particular, H_t is $\mathfrak{E}_0$ if $Y_t = \mathfrak{E}_0$, is $\mathfrak{E}_1$ if $Y_t = \mathfrak{E}_1$, and is 1 if $Y_t \in \mathbb{R}$. We use h_t to denote the realization of H_t.

The time-evolutions of π_t^1 and π_t^2 is similar to Lemma 2. In particular, there exists a function F^2 such that

$$\pi_t^2 = F^2(\pi_t^1, \varphi_t, h_t) = \begin{cases} \delta_0, & \text{if } h_t = 1 \\ \pi_t^1|_{\varphi_t}, & \text{if } h_t = \mathfrak{E}_1 \\ \pi_t^1, & \text{if } h_t = \mathfrak{E}_0. \end{cases} \tag{18}$$

Consequently, the dynamic program of Theorem 1 is now given by

$$V_{T+1}^1(s, \pi^1) = 0, \tag{19}$$

and for $t \in \{T, \dots, 0\}$

$$\begin{aligned} V_t^1(s, \pi^1) = \min_{\varphi \colon \mathbb{R}\to\{0,1\}} \Big\{ &\lambda \pi^1(B_1(\varphi)) \\ &+ W_t^0(\pi^1, \varphi)\pi^1(B_0(\varphi)) + W_t^1(\pi^1, \varphi)\pi^1(B_1(\varphi)) \Big\}, \end{aligned} \tag{20}$$

$$V_t^2(s, \pi^2) = D(\pi^2) + V_{t+1}^1(s, F^1(\pi^2)), \tag{21}$$

where,

$$\begin{aligned} W_t^0(\pi^1, \varphi) &= Q_{s0}V_t^2(0, \pi^1) + Q_{s1}V_t^2(1, \pi^1|_\varphi), \\ W_t^1(\pi^1, \varphi) &= Q_{s0}V_t^2(0, \pi^1) + Q_{s1}V_t^2(1, \delta_0), \\ D(\pi^2) &= \min_{\hat{e}\in\mathbb{R}} \int_{\mathbb{R}} d(e - \hat{e})\pi^2(e)de. \end{aligned}$$

Note that due to the change of variables, the expression for W_t^1 does not depend on the transmitted symbol. Consequently, the expression for V_t^1 is simpler than that in Theorem 1.

Definition 1 (Symmetric and unimodal density) *A probability density function π on reals is said to be* symmetric and unimodal (SU) *around $c \in \mathbb{R}$ if for any $x \in \mathbb{R}$, $\pi(c - x) = \pi(c + x)$ and π is non-decreasing in the interval $(-\infty, c]$ and non-increasing in the interval $[c, \infty)$.*

Definition 2 (Threshold based prescription) *Given $c \in \mathbb{R}$, a prescription $\varphi \colon \mathbb{R} \to \{0,1\}$ is called* threshold based around *c if there exists $k \in \mathbb{R}$ such that $\varphi(e) = 1$ if $|e - c| \geq k$, else $\varphi(e) = 0$.*

Let $\mathcal{F}(c)$ denote the family of all threshold-based prescription around c.

We first prove a weaker version of the structure of optimal transmission strategies. In particular, there exist threshold functions $\tilde{k}_t \colon \{0,1\} \times \Delta(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ such that the following transmission strategy is optimal:

$$f_t(E_t, S_{t-1}, \Pi_t^1) = \begin{cases} 1, & \text{if } |E_t| \geq \tilde{k}_t(S_{t-1}, \Pi_t^1) \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

We prove (22) by induction. Note that $\pi_0^1 = \delta_0$ which is $\mathrm{SU}(0)$. Therefore, by [22, Lemma 3, (P2)], there exists a threshold-based prescription $\varphi_0 \in \mathcal{F}(0)$ that is optimal. This forms the basis of induction. Now assume that until time $t-1$, all prescriptions are in $\mathcal{F}(0)$. By [22, Properties 2 and 3], Π_t^1 is $\mathrm{SU}(0)$. Therefore, by [22, Lemma 3, (P2)], there exists a threshold-based prescription $\varphi_t \in \mathcal{F}(0)$ that is optimal. This proves the induction step and, hence, by the principle of induction, threshold-based prescriptions of the form (22) are optimal for all time.

Observe that [22, Properties 2 and 3] also imply that for all t, Π_t^2 is $\mathrm{SU}(0)$. Therefore, by [22, Property 1], the optimal estimate $\hat{E}_t = 0$. Recall that $\hat{E}_t = \hat{X}_t - Z_t$. Thus, $\hat{X}_t = Z_t$. This proves the first part of Theorem 2.

To prove that there exist optimal transmission strategies where the thresholds do not depend on Π_t^1, we fix the estimation strategy to be of the form (16) and consider the problem of finding the best transmission strategy at the sensor. This is a single-agent (centralized) stochastic control problem and the optimal solution is given by the following dynamic program: $J_{T+1}(e, s) = 0$ and for $t \in \{T, \dots, 0\}$,

$$J_t(e, s) = \min\{J_t^0(e, s), J_t^1(e, s)\} \tag{23}$$

where

$$\begin{aligned} J_t^0(e, s) &= d(e) + Q_{s0}\mathbb{E}_W[J_{t+1}(ae + W, 0)] \\ &\quad + Q_{s1}\mathbb{E}_W[J_{t+1}(ae + W, 1)], \quad (24) \\ J_t^1(e, s) &= \lambda + Q_{s0}d(e) + Q_{s0}\mathbb{E}_W[J_{t+1}(ae + W, 0)] \\ &\quad + Q_{s1}\mathbb{E}_W[J_{t+1}(W, 1)], \quad (25) \end{aligned}$$

Using ideas similar to [25], one can show that the threshold based transmission strategy (17) is optimal.

IV. Conclusion

In this paper, we identify the structure of optimal transmission and estimation strategies for remote estimation over Gilbert-Elliot channel with feedback. When the source is first-order autoregressive process with symmetric and unimodal noise distribution, the optimal strategy has a threshold structure and the optimal estimation strategy is Kalman-like. A natural question is how to determine the optimal thresholds. For finite horizon setup, these can be determined using the dynamic program of (23)–(25). For infinite horizon setup, we expect that the optimal threshold will not depend on time. We believe that it should be possible to evaluate the performance of a generic threshold based strategy using an argument similar to the renewal theory based argument presented in [9] for channels without packet drops.

The argument in [9] relied on the distortion and time until the first successful reception, which were computed using Fredholm integral equations of the second time. However, in the current model, the corresponding Fredholm integral equations will be difficult to solve numerically. We believe that it would be possible to adapt the sampling based stochastic approximation techniques of [10] to the current setup.

References

[1] J. Marschak, "Towards an economic theory of organization and information," *Decision processes*, vol. 3, no. 1, pp. 187–220, 1954.

[2] Y. Xu and J. P. Hespanha, "Optimal communication logics in networked control systems," in *Proceedings of 43rd IEEE Conference on Decision and Control*, vol. 4, 2004, pp. 3527–3532.

[3] O. C. Imer and T. Başar, "Optimal estimation with limited measurements," *Joint 44the IEEE Conference on Decision and Control and European Control Conference*, vol. 29, pp. 1029–1034, 2005.

[4] M. Rabi, G. Moustakides, and J. Baras, "Adaptive sampling for linear state estimation," *SIAM Journal on Control and Optimization*, vol. 50, no. 2, pp. 672–702, 2012.

[5] G. M. Lipsa and N. C. Martins, "Remote state estimation with communication costs for first-order LTI systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2013–2025, 2011.

[6] A. Nayyar, T. Başar, D. Teneketzis, and V. V. Veeravalli, "Optimal strategies for communication and remote estimation with an energy harvesting sensor," *IEEE Transactions on Automatic Control*, vol. 58, no. 9, pp. 2246–2260, 2013.

[7] A. Molin and S. Hirche, "An iterative algorithm for optimal event-triggered estimation," in *4th IFAC Conference on Analysis and Design of Hybrid Systems (ADHS'12)*, 2012, pp. 64–69.

[8] D. Shi, L. Shi, and T. Chen, *Event-Based State Estimation: A Stochastic Perspective*. Springer, 2015, vol. 41.

[9] J. Chakravorty and A. Mahajan, "Fundamental limits of remote estimation of autoregressive markov processes under communication constraints," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1109–1124, March 2017.

[10] ——, "Remote state estimation with packet drop," in *6th IFAC Workshop on Distributed Estimation and Control in Networked Systems*, Sep 2016.

[11] X. Ren, J. Wu, K. H. Johansson, G. Shi, and L. Shi, "Infinite horizon optimal transmission power control for remote state estimation over fading channels," *arxiv: 1604.08680v1*, Apr. 2016.

[12] W. Chen, J. Wang, D. Shi, and L. Shi, "Event-based state estimation of hidden markov models through a Gilbert-Elliott channel," *to appear in IEEE Transactions on Automatic Control*, 2017.

[13] X. Gao, E. Akyol, and T. Başar, "Optimal sensor scheduling and remote estimation over an additive noise channel," in *American Control Conference (ACC)*, 2015, pp. 2723–2728.

[14] ——, "Optimal estimation with limited measurements and noisy communication," in *2015 54th IEEE Conference on Decision and Control (CDC)*, Dec 2015, pp. 1775–1780.

[15] ——, "On remote estimation with multiple communication channels," in *2016 American Control Conference (ACC)*, July 2016, pp. 5425–5430.

[16] H. S. Witsenhausen, "On the structure of real-time source coders," *BSTJ*, vol. 58, no. 6, pp. 1437–1451, July-August 1979.

[17] J. C. Walrand and P. Varaiya, "Optimal causal coding-decoding problems," *IEEE Trans. Inf. Theory*, vol. 29, no. 6, pp. 814–820, Nov. 1983.

[18] D. Teneketzis, "On the structure of optimal real-time encoders and decoders in noisy communication," *IEEE Trans. Inf. Theory*, pp. 4017–4035, Sep. 2006.

[19] A. Mahajan and D. Teneketzis, "Optimal design of sequential real-time communication systems," *IEEE Trans. Inf. Theory*, vol. 55, no. 11, pp. 5317–5338, Nov. 2009.

[20] E. N. Gilbert, "Capacity of a burst-noise channel," *Bell System Technical Journal*, vol. 39, no. 5, pp. 1253–1265, 1960.

[21] E. O. Elliott, "Estimates of error rates for codes on burst-noise channels," *Bell System Technical Journal*, vol. 42, no. 5, pp. 1977–1997, 1963.

[22] J. Chakravorty and A. Mahajan, "Structure of optimal strategies for remote estimation over Gilbert-Elliott channel with feedback," *arXiv: 1701.05943*, Jan 2017.

[23] A. Nayyar, A. Mahajan, and D. Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," *IEEE Trans. Autom. Control*, vol. 58, no. 7, pp. 1644–1658, jul 2013.

[24] S. Yuksel, "On optimal causal coding of partially observed Markov sources in single and multiterminal settings," *IEEE Trans. Inf. Theory*, vol. 59, no. 1, pp. 424–437, 2013.

[25] J. Chakravorty and A. Mahajan, "Sufficient conditions for the value function and optimal strategy to be even and quasi-convex," *arXiv: 1703.10746*, Mar 2017.