Structural results for two-user interactive communication

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***Abstract*—In this paper we consider an interactive communication system with two users, who sequentially observe two correlated sources, and send the quantized observation symbol to each other. The sources are functions of a random variable, which the users wish to estimate. The transmission is costly and the fidelity of reconstruction is measured by a distortion function. We model this problem using dynamic team theory. The two users are viewed as two decision makers that have access to different information but need to coordinate their actions to minimize a common objective. Through a series of simplifications, we identify time-homogeneous information states (sufficient statistics) for the encoding and decoding strategies and a dynamic programming decomposition to compute the optimal strategies.**

I. Introduction

In recent years, there has been an increasing interest in interactive communication in the context interactive computing and interactive source coding.

In interactive computing (or communication complexity), which was introduced in [1], two users who have access to different random variables want to compute a function of these random variables. Communication complexity refers to the minimum number of bits the two users must communicate to compute the function with high reliability. See [2], [3] for detailed overview.

In interactive source coding, each user is interested in reproducing the source outputs, either exactly or with loss, of the other user [4], [5]. Interactive source coding for function computation is considered in [6]–[8].

Interactive communication is also important in information theorectic security for generating secret keys [9]. See [10], [11] for detailed overview.

Most of the above models assume that the variable of interest is a function of the observations of the two users. Moreover, it is assumed that the users observe either a single random variable or a large block of random variables at once.

In this paper, we consider a model where the variable of interest is not observed by either of the users. Rather, the users *sequentially* obtain noisy observations of a static random variable. At each time, after making its observation, user 1 sends a quantized symbol to user 2; after receiving user 1's symbol and its own observation, user 2 sends a quantized symbol to user 1. Then both users generate an estimate of the underlying static random variable. This process repeats over a finite time horizon. At each stage, the users quantize and estimate based on the history of their source observations and the quantized symbols from the other user. The per-step cost consists of two parts: a cost associated with each quantized symbol and a distortion cost between the underlying random variable and the estimate made by the two users at that time. The objective is to minimize the total expected cost over a finite horizon.

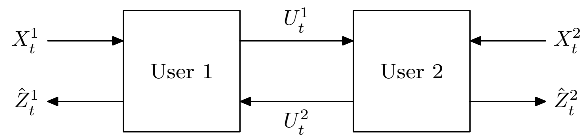

Fig. 1. Block diagram of an interactive communication system.

The above model is related to real-time (or zero-delay) communication. Real-time source coding was considered in [12]; joint source channel coding with noiseless feedback was considered in [13]; joint source channel coding without feedback was considered in [14], [15].

Notation

Random variables are denoted by uppercase letters, e.g., X, Y; their realizations are denoted by corresponding lowercase letters, e.g., x, y. $x_{1:t}$ is a short hand for the vector $(x_1, \dots, x_t)$. x_t is a short hand for the vector $(x_t^1, x_t^2 \dots)$. $\mathbb{P}(\cdot)$ denotes the probability of an event, $\mathbb{E}[\cdot]$ denotes the expectation of a random variable, and $\mathbb{1}\{\cdot\}$ denotes the indicator function of a statement. For a set $\mathcal{X}$, $\Delta(\mathcal{X})$ denotes set of all probability distributions on $\mathcal{X}$.

II. Model and problem formulation

Consider an interactive communication system as shown in Fig. 1. The system consists of two users that observe correlated sources $\{X_t^i\}_{t=1}^{\infty}$, $X_t^i \in \mathcal{X}^i$, $i \in \{1, 2\}$. The sources are generated according to

$$X_t^i = h_t^i(Z, W_t^i), \tag{1}$$

where h^i is a known function, $Z \in \mathcal{Z}$ is a random variable of interest and $\{W_t^1\}_{t=1}^{\infty}$, $\{W_t^2\}_{t=1}^{\infty}$ are i.i.d. sequence that are independent of each other and also independent of Z. $\{X_t^1\}_{t=1}^{\infty}$, $\{X_t^2\}_{t=1}^{\infty}$ are correlated across time and also correlated with each other. For ease of exposition, we assume that the alphabets $\mathcal{Z}$, $\mathcal{X}^1$, and $\mathcal{X}^2$ are finite.

The users sequentially quantize their observations and send a symbol to the other user over a finite-rate noiseless channel. In particular, during time slot t, first user 1 sends a symbol $U_t^1 \in \mathcal{U}^1$ to user 2, then user 2 sends a symbol $U_t^2 \in \mathcal{U}^2$ to user 1. Both $\mathcal{U}^1$ and $\mathcal{U}^2$ are finite sets and the quantized symbols are generated based on all the information available to users, i.e.,

$$U_t^1 = f_t^1(X_{1:t}^1, U_{1:t-1}^1, U_{1:t-1}^2),\ U_t^2 = f_t^2(X_{1:t}^2, U_{1:t}^1, U_{1:t-1}^2).$$

where f_t^i is called the *encoding rule* of user i at time t. Cost functions $c^i\colon \mathcal{U}^i \to \mathbb{R}_{\geq 0}$ measure the cost of transmission.[1]

During time slot t, after observing the quantized symbol from user 1, user 2 generates an estimate $\hat{Z}_t^2 \in \mathcal{Z}$; after observing the quantized symbol from user 2, user 1 generates an estimate $\hat{Z}_t^1 \in \mathcal{Z}$. These estimates are generated based on all the information available to the users, i.e.,

$$\hat{Z}_t^1 = g_t^1(X_{1:t}^1, U_{1:t}^1, U_{1:t-1}^2),\ \hat{Z}_t^2 = g_t^2(X_{1:t}^2, U_{1:t}^1, U_{1:t}^2), \quad (2)$$

where g_t^i is called the *decoding rule* of user i at time t. Distortion functions $d_t^i\colon \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_{\geq 0}$ measure the fidelity of reconstruction at time t

The sequence $\mathbf{f}^i := (f_1^i, \cdots, f_T^i)$, $i \in \{1, 2\}$ is called the *encoding strategy* of user i. Similarly, the sequence $\mathbf{g}^i := (g_1^i, \cdots, g_T^i)$, $i \in \{1, 2\}$ is called the *decoding strategy* of user i. The tuple $(\mathbf{f}^1, \mathbf{f}^2, \mathbf{g}^1, \mathbf{g}^2)$ is called the *communication strategy*.

The performance $J(\mathbf{f}^1, \mathbf{f}^2, \mathbf{g}^1, \mathbf{g}^2)$ of a communication strategy $(\mathbf{f}^1, \mathbf{f}^2, \mathbf{g}^1, \mathbf{g}^2)$ is given by the expected total transmission cost and distortion under that strategy, i.e.,

$$J(\mathbf{f}^1, \mathbf{f}^2, \mathbf{g}^1, \mathbf{g}^2) = \mathbb{E}\Big[\sum_{t=1}^{T}\sum_{i=1}^{2}\big[c^i(U_t^i) + d_t^i(Z, \hat{Z}_t^i)\big]\Big], \quad (3)$$

where the expectation is with respect to a joint measure on all system variables induced by the choice of $(\mathbf{f}^1, \mathbf{f}^2, \mathbf{g}^1, \mathbf{g}^2)$.

We are interested in the following optimization problem.

Problem 1 *For the interactive communication system described above, choose a communication strategy* $(\mathbf{f}^1, \mathbf{f}^2, \mathbf{g}^1, \mathbf{g}^2)$ *that minimizes total expected cost* $J(\mathbf{f}^1, \mathbf{f}^2, \mathbf{g}^1, \mathbf{g}^2)$ *defined in* (3).

A key feature of the above model is that both users must generate an estimate of Z at each step. This feature makes our model different from the standard model of interactive communication, where there are multiple rounds of communication and each user generates a single estimate at the end of communication.

Due to this sequential nature of estimation, the standard information theoretic arguments cannot be used. Instead, we directly analyze the optimization problem. The above optimization problem has two decision makers—user 1 and user 2—that have access to different information but need to cooperate and coordinate their actions to minimize a common objective. Therefore, it belongs to the category of *dynamic team* problems [21].

The main conceptual difficulty in solving the above optimization problem is that the information available at both users is increasing with time, and hence, so is the domain of their stratgies. For example, suppose all alphabets are binary. Then there are $2^{2^{3t-2}}$ possibilities for encoding and decoding strategies at each user at time t. Thus, even for a horizon of 3, there are about 10^{175} possible communication strategies (with the dominant term being $(2^{2^7})^4$ possibilities at stage 3). Thus, a brute force search is computationally intractable.

In single agent muti-stage optimization problem, such a difficulty is resolved by identifying a time-homogeneous information state at the decision maker. It is difficult to identify such information states in multi-agent multi-stage decision problems because the different decision makers have access to different information.

We resolve this difficulty in two steps using ideas from team theory. In the first step, we take a person-by-person approach. We arbitrarily fix the strategy of one user, say user 2, and search for the *best response* strategy at user 1. By showing that $X_{1:t}^1$ and $X_{1:t}^2$ are conditionally independent given $(Z, U_{1:t}^1, U_{1:t}^2)$, we identify a sufficient statistic $\xi_{t|t-1}^i$ (to be defined later) of $x_{1:t}^i$. This means that there is no loss of optimality in restricting attention to encoders of the form:

$$U_t^1 = \hat{f}_t^1(\Xi_{t|t-1}^1, U_{1:t-1}^1, U_{1:t-1}^2), U_t^2 = \hat{f}_t^2(\Xi_{t|t-1}^2, U_{1:t}^1, U_{1:t-1}^2).$$

A similar structure for the decoders is also identified.

In the second step, we use the common-information approach of [22] and identify a sufficient statistic π_t^1 (to be defined later) of $(u_{1:t-1}^1, u_{1:t-1}^2)$ at user 1 and a sufficient statistic π_t^2 (to be defined later) of $(u_{1:t}^1, u_{1:t-1}^2)$ at user 2. This means that there is no loss of optimality in restricting attention to encoders of the form:

$$U_t^1 = \tilde{f}_t^1(\Xi_{t|t-1}^1, \Pi_t^1), \quad U_t^2 = \tilde{f}_t^2(\Xi_{t|t-1}^2, \Pi_t^2).$$

We also identify a dynamic program that determines optimal encoding and decoding strategies of the above form.

III. The Main Results

A. A conditional independence result

The sources are conditionally independent given Z. Our main results rely on the fact that the sources remain conditionally independent when conditioned on Z and the communicated symbols. For ease of notation, $\mathbb{P}(X_{1:t} = x_{1:t}^1 \mid Z = z, U_{1:t}^1 = u_{1:t}^1, U_{1:t}^2 = u_{1:t}^2)$ is denoted by $\mathbb{P}(x_{1:t}^1 \mid z, u_{1:t}^1, u_{1:t}^2)$. We use similar notation for other probability expressions as well.

[1]Assuming a transmission cost allows us to model different scenarios. For example, in variable rate communication, the cost function $c^i(u^i) = \log |u^i|$ is used (see [20]). Even in fixed rate communication, a user may not transmit at each time and the transmission cost is zero for not transmitting and a constant for transmitting.

Lemma 1 *For any arbitrary encoding strategies* $(\mathbf{f}^1, \mathbf{f}^2)$ *and any realization* z *of* Z, $x^i_{1:t}$ *of* $X^i_{1:t}$, *and* $u^i_{1:t} \in U^i_{1:t}$, $i \in \{1, 2\}$, *we have the following:*

$$\begin{aligned}&\mathbb{P}(x^1_{1:t}, x^2_{1:t} \mid z, u^1_{1:t}, u^2_{1:t}) \\ &\quad = \mathbb{P}(x^1_{1:t} \mid z, u^1_{1:t}, u^2_{1:t})\mathbb{P}(x^2_{1:t} \mid z, u^1_{1:t}, u^2_{1:t})\end{aligned} \tag{4}$$

and

$$\begin{aligned}&\mathbb{P}(x^1_{1:t}, x^2_{1:t} \mid z, u^1_{1:t-1}, u^2_{1:t-1}) \\ &\quad = \mathbb{P}(x^1_{1:t} \mid z, u^1_{1:t-1}, u^2_{1:t-1})\mathbb{P}(x^2_{1:t} \mid z, u^1_{1:t-1}, u^2_{1:t-1})\end{aligned} \tag{5}$$

Lemma 1 can be proved using algebraic calculations involving chain rule of probability and total probability. The details of the proof is omitted here due to the limitation of space. Similar results on conditional independence is discussed in [23] (for decentralized control systems with control sharing), in [24], [25] (for secret key argument and secure computing) and in [26] (for CEO problems).

B. Belief states and their update

For ease of notation, define $U_t = (U^1_t, U^2_t)$.

Definition 1 *For any realization* $x^i_{1:t}$ *of* $X^i_{1:t}$ *and* $u^i_{1:t}$ *of* $U^i_{1:t}$, $i \in \{1, 2\}$ *define belief states* $\xi^i_{t|t-1}, \xi^i_{t|t} \in \Delta(\mathcal{Z})$ *as follows: for any* $z \in \mathcal{Z}$,

$$\begin{aligned}\xi^i_{t|t-1}(z) &= \mathbb{P}(Z = z \,|\, X^i_{1:t} = x^i_{1:t}, U_{1:t-1} = u_{1:t-1}), \\ \xi^i_{t|t}(z) &= \mathbb{P}(Z = z \,|\, X^i_{1:t} = x^i_{1:t}, U_{1:t} = u_{1:t}),\end{aligned}$$

where $u_t = (u^1_t, u^2_t)$.

$\xi^i_{t|t-1}$ denote user i's belief on Z after it has observed the source realization of time t but before the communication of that time slot takes place; $\xi^i_{t|t}$ denotes the belief after the communication has taken place. For a specific realization of $(x^i_{1:t}, u_{1:t})$, $\xi^i_{t|t-1}$ and $\xi^i_{t|t}$ are probability distributions. When the conditioning is on random variables $(X^i_{1:t}, U_{1:t})$, the beliefs are $\Delta(\mathcal{Z})$ valued random variables that we denote by the corresponding uppercase letters $\Xi^i_{t|t-1}$ and $\Xi^i_{t|t}$.

In order to derive the structural results, it is important to identify how these beliefs depend on the strategy. To do so, we determine how the beliefs evolve with time. In the sequel, we use $-i$ to denote the user different from user i.

Lemma 2 *There exist functions* $F^i_{t|t}$, $F^i_{t+1|t}$, $i \in \{1, 2\}$, *such that*

$$\xi^i_{t|t} = F^i_{t|t}\big(\xi^i_{t|t-1}, u_{1:t}, \mathbf{f}^{-i}\big), \; \xi^i_{t+1|t} = F^i_{t+1|t}\big(\xi^i_{t|t}, u_{1:t}, x^i_{t+1}\big). \tag{6}$$

By combining these two, we get that there exists a function F^i_t *such that*

$$\xi^i_{t+1|t+1} = F^i_t(\xi^i_{t|t}, u_{1:t}, x^i_{t+1}, \mathbf{f}^{-i}). \tag{7}$$

The proof of Lemma 2 is given in the Appendix.

C. Step 1: The person-by-person approach

As explained earlier, we follow a two-step approach to derive the structure of optimal strategies. In the first step, we follow a person-by-person approach. We arbitrarily fix the strategy of one user and then investigate the best response strategy at the other user.

First, we identify the structure of optimal decoding strategies. Since decoding is a filtering problem, we have:

Proposition 1 (Structure of optimal decoding strategies) *There is no loss of optimality to restrict the attention to decoding strategies of the form:*

$$\hat{Z}^i_t = \hat{g}^i(\Xi^i_{t|t}), \quad i \in \{1, 2\}, \tag{8}$$

where $\hat{g}^i_t$ *is given by*

$$\hat{g}^i(\xi^i) = \arg\min_{\hat{z}^i \in \mathcal{Z}} \sum_{z \in \mathcal{Z}} d^i(z, \hat{z}^i)\xi^i(z).$$

Now, we fix the decoders at both users according to (8) and find the *best response* encoder. By combining Lemmas 1 and 2, we show the following:

Lemma 3 *Fix decoding strategies* $\mathbf{g}^1, \mathbf{g}^2$ *to be of the form* (8). *Arbitrarily fix the communication strategy* $\mathbf{f}^2$ *of user 2. Then,* $R^1_t = (\Xi^i_{t|t-1}, U_{1:t-1})$ *is an information state for the encoder at user 1. In particular,* R^1_t *satisfies the following properties:*

1) R^1_t *is a function of the information* $(X^1_{1:t}, U_{1:t-1})$ *available at user 1.*
2) *The conditional distribution of* R^1_{t+1} *given all the available information* $(X^1_{1:t}, U_{1:t-1})$ *and the current action* U^1_t *depends only on* R^1_t *and* U^1_t, *i.e.,*
$$\mathbb{P}(R^1_{t+1} \,|\, X^1_{1:t}, U_{1:t-1}, U^1_t) = \mathbb{P}(R^1_{t+1} \,|\, R^1_t, U^1_t). \tag{9}$$
3) R^1_t *is a sufficient statistic for the current cost. In particular,*
$$\begin{aligned}&\mathbb{E}\Big[\sum_{i\in\{1,2\}} (c^i(U^i_t) + d^i(Z, \hat{Z}^i_t)) \,|\, X^1_{1:t}, U_{1:t-1}, U^1_t\Big] \\ &= \mathbb{E}\Big[\sum_{i\in\{1,2\}} (c^i(U^i_t) + d^i(Z, \hat{Z}^i_t)) \,|\, R^1_t, U^1_t\Big]\end{aligned} \tag{10}$$

The proof of Lemma 4 involves some standard algebraic calculations, and is skipped here due to limitation of space.

A similar result holds if $(\mathbf{f}^1, \mathbf{g}^1)$ is fixed and we consider the best response at user 2.

Lemma 3 implies that $\{R^1_t\}_{t\geq 1}$ is a controlled Markov process with control action U^1_t. Therefore, there is no loss of optimality to restrict attention to *Markov strategies*

$$U^1_t = \hat{f}^1_t(\Xi^1_{t|t-1}, U_{1:t-1}).$$

By repeating the argument at user 2, we get the following:

Proposition 2 (Structure of optimal encoding strategies) *There is no loss of optimality to restrict the attention to encoding strategies of the form:*

$$U^1_t = \hat{f}^1_t(\Xi^1_{t|t-1}, U_{1:t-1}), \; U^2_t = \hat{f}^2_t(\Xi^2_{t|t-1}, U_{1:t-1}, U^1_t). \tag{11}$$

D. Step 2: The common-information approach

We have identified the structure of optimal decoders in closed form and simplified the structure of optimal encoders. In this section, we refine the structural result of Proposition 2 by following the common-information approach of [22].

We fix the decoding strategies as specified in Proposition 1 and consider the problem of optimally selecting encoding strategies that are of the form (11). Following [22], define the *common information* to be the data that is observed by all future decision makers, i.e., define the common information C_t^i at user i at time t as:

$$C_t^1 = U_{1:t-1}, \quad C_t^2 = (U_{1:t-1}, U_t^1).$$

Define the remaining information at user i as local information L_t^i, i.e., $L_t^i = \xi_{t|t-1}^i$. Thus, we can say

$$U_t^i = f_t^i(L_t^i, C_t^i).$$

The main idea of [22] is to consider Problem 1 from the point of view of a virtual decision maker that observes C_t^i and chooses prescriptions $\phi_t^i : L_t^i \mapsto U_t^i$ that map local information to actions. The encoders simply use these mappings and their local information to generate U_t^i.

It is shown in [22] that the above *coordinated system* is equivalent to the original system. Since the coordinated system has only one decision maker, it can be solved using tools from Markov decision theory. To describe the results, we first note that:

Lemma 4 *For the encoders are of the form given in Proposition 2, the update of Lemma 2 can be written as*

$$\xi_{t|t}^i = F_{t|t}^i\big(\xi_{t|t-1}^i, u_t^{-i}, \phi_t^{-i}\big). \tag{12}$$

Definition 2 *For any realization $x_{1:t}^i$ of $X_{1:t}^i$ and $u_{1:t}^i$ of $U_{1:t}^i$, $i \in \{1,2\}$ define belief states $\pi_t^i \in \Delta(\Delta(\mathcal{X}^1) \times \Delta(\mathcal{X}^2))$ as follows: for any $\xi_{t|t-1}^1, \xi_{t|t-1}^2 \in \Delta(\mathcal{Z})$,*

$$\pi_t^1(\xi^1, \xi^2) = \mathbb{P}(\Xi_{t|t-1}^1 = \xi^1, \Xi_{t|t-1}^2 = \xi^2 \,|\, U_{1:t-1} = u_{1:t-1}),$$
$$\pi_t^2(\xi^1, \xi^2) = \mathbb{P}(\Xi_{t|t-1}^1 = \xi^1, \Xi_{t|t-1}^2 = \xi^2 \,|\, U_{1:t-1} = u_{1:t-1}, U_t^1 = u_t^1),$$

where $u_t = (u_t^1, u_t^2)$.

Then, similar to Lemma 4, we can show the following

Lemma 5 *There exist functions $\tilde{F}_t^i$, $i \in \{1,2\}$, such that*

$$\pi_{t+1}^1 = \tilde{F}_t^1\big(\pi_t^2, U_t^2, \phi_t^2\big), \quad \pi_t^2 = \tilde{F}_t^2\big(\pi_t^1, U_t^1, \phi_t^1\big). \tag{13}$$

According to the discussion above, we fix the decoding strategy to be of the form Proposition 1 and restrict encoding strategy to be of the form Proposition 2. The optimization problem then satisfies the partial history sharing model of [22]. Therefore, from [22], we get the following:

Theorem 1 *There is no loss of optimality in restricting attention to encoding strategies of the form:*

$$U_t^1 = \tilde{f}_t^1(\xi_{t|t-1}^1, \Pi_t^1), \quad U_t^2 = \tilde{f}_t^2(\xi_{t|t-1}^2, \Pi_t^2).$$

Moreover, optimal strategies of this form may be determined from the following dynamic program. Define

$$D_t^i(\xi_{t|t}^i) = \sum_{z \in \mathcal{Z}} d_t^i(z, \hat{g}^i(\xi_{t|t}^i))\xi_{t|t}^i(z), \; i \in \{1,2\}.$$

Then, recursively define value functions $\{V_t^1\}_{t \geq 1}$ and $\{V_t^2\}_{t \geq 1}$ as follows:

$$V_{T+1}^2(\pi^2) = 0 \tag{14}$$

and for $t = T, T-1, \ldots, 1$

$$V_t^2(\pi^2) = \min_{\phi_t^2 : \Delta(\mathcal{Z}) \to \mathcal{U}^2} \mathbb{E}[c^2(U_t^2) + D_t^2(\Xi_{t|t}^2) + V_{t+1}^1(\Pi_{t+1}^1) \,|\, \Pi_t^2 = \pi^2, U_t^2 = \phi_t^2(\Xi_{t|t-1}^2)], \tag{15}$$

and

$$V_t^1(\pi^1) = \min_{\phi_t^1 : \Delta(\mathcal{Z}) \to \mathcal{U}^1} \mathbb{E}[c^1(U_t^1) + D_t^1(\Xi_{t|t}^1) + V_t^2(\Pi_t^2) \,|\, \Pi_t^1 = \pi^1, U_t^1 = \phi_t^1(\Xi_{t|t-1}^1)]. \tag{16}$$

Let $\psi_t^2(\pi^2)$ denote the arg min of (15) and $\psi^1(\pi^1)$ denote the arg min of (16). Then, the optimal strategy $\tilde{\mathbf{f}}^1, \tilde{\mathbf{f}}^2$ is given by

$$\tilde{f}_t^i(\xi_{t|t-1}^i, \pi_t^i) = \psi_t^i(\pi_t^i)(\xi_{t|t-1}^i). \tag{17}$$

Note that the expectations in (16) and (15) can be computed using the update rules in Lemmas 2 and 5.

IV. Discussion and conclusion

Theorem 1 identifies a sufficient statistic at the encoder and the decoder; the domain of which does not depend on time. Moreover, the dynamic program provides a way to identify optimal (or sub-optimal) strategies. As a consequence, the search complexity increases linearly with time horizon (rather than double exponentially, as for brute force search). If $\mathcal{Z}$ is finite, say of cardinality n, then $\Delta(\mathcal{X}^i)$ may be viewed as an element of $\mathbb{R}^{n-1}$; and hence the belief space is the space of probability distributions on $\mathbb{R}^{2n-2}$.

Note that the dynamic program is similar to the dynamic programs for partially observable Markov decision processes (POMDP). So, it is possible to use point-based algorithms for continuous state POMDPs to numerically solve the resultant dynamic program. Another option is to use discretization based algorithms developed for real-time communication [27]. Following [28], it may be possible to establish that threshold-based strategies are optimal when all random variables are Gaussian and the transmitter has the option of not transmitting.

Since the domain of the encoding and decoding strategies is not changing with time, the result of Theorem 1 naturally extends to infinite horizon setups as well. We expect that under appropriate regularity conditions, the optimal strategy is time homogeneous and given by the fixed point of a dynamic program. It may be possible to use such a dynamic program to find bounds on time average distortion.

Although the results of this paper were derived for a two-user interactive communication system, they generalize to the following multi-terminal setup. Consider n users with observations similar to (1). During time-slot t, first user 1 broadcasts a symbol U_t^1 to all users. Then user 2 broadcasts U_t^2 to all users, and so on, until user n broadcasts U_t^n to all users. All users generate an estimate of Z and the process repeats at $t+1$. Such a multi-user setup can be analyzed using the same approach as presented in this paper.

ACKOWLEDGEMENT

This work was supported in part by Fonds de recherche du Québec–Nature et technologies (FRQNT) Team Grant PR-173396. The authors thank an anonymous reviewer for pointing out an error in the previous version of Lemma 2.

APPENDIX

Proof of Lemma 2: Let us arbitrarily fix the strategy of user 2. Consider the following:

$$\xi^1_{t|t}(z) = \mathbb{P}(z \mid x^1_{1:t}, u_{1:t}) = \frac{\mathbb{P}(z, x^1_{1:t}, u_{1:t})}{\mathbb{P}(x^1_{1:t}, u_{1:t})}.$$

Now, by total probability we get,

$$\mathbb{P}(z, x^1_{1:t}, u_{1:t}) = \mathbb{P}(u^2_t \mid z, x^1_{1:t}, u^1_{1:t}, u^2_{1:t-1})\xi^1_{t|t-1}(z)$$
$$\mathbb{P}(u^2_t \mid z, x^1_{1:t}, u^1_{1:t}, u^2_{1:t-1})$$
$$= \sum_{x^2_{1:t}} \mathbb{1}\{U^2_t = f^2_t(x^2_{1:t}, u_{1:t-1}, u^1_t)\}\mathbb{P}(x^2_{1:t} \mid z, u_{1:t-1}, u^1_t)$$

Note that the RHS of the last equation depends on the communication strategy $\mathbf{f}^1, \mathbf{f}^2, \mathbf{g}^1, \mathbf{g}^2$ only through $\mathbf{f}^2$. Also, by total probability,

$$\mathbb{P}(x^1_{1:t}, u_{1:t}) = \sum_z \sum_{x^2_{1:t}} \mathbb{P}(u^2_t \mid z, x^2_{1:t}, u^1_{1:t}, u^2_{1:t-1})\mathbb{P}(x^2_{1:t} \mid z, u_{1:t-1}).$$

Substituting this in the expression for $\xi^1_{t|t}$ and simplifying, we get that $\xi^1_{t|t}$ is a function of $(\xi^1_{t|t-1}, u_{1:t}, \mathbf{f}^2)$

Now consider the following,

$$\xi^1_{t+1|t}(z) = \mathbb{P}(z \mid x^1_{1:t+1}, u_{1:t}) = \frac{\mathbb{P}(z, x^1_{t+1} \mid x^1_{1:t}, u_{1:t})}{\mathbb{P}(x^1_{t+1} \mid x^1_{1:t}, u_{1:t})}. \quad (18)$$

Also, it can be shown by similar calculation that $\mathbb{P}(z, x^1_{t+1} \mid x^1_{1:t}, u_{1:t}) = \mathbb{P}(x^1_{t+1} \mid z)\xi^1_{t|t}(z)$. Substituting back in (18), we have

$$\xi^1_{t+1|t}(z) = \frac{\mathbb{P}(x^1_{t+1} \mid z, u_{1:t})\xi^1_{t|t}}{\sum_z \mathbb{P}(x^1_{t+1} \mid z, u_{1:t})\xi^1_{t|t}(z)} =: F^1_{t+1|t}(\xi^1_{t|t}, u_{1:t}, x^1_{t+1}).$$

This completes the proof for user 1. The results for user 2 can be derived similarly.

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