Fixed Delay Optimal Joint Source-Channel Coding for Finite-Memory Systems

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***Abstract*— We consider zero-delay or fixed finite-delay joint source channel coding of Markov sources using finite memory encoder and decoder. The objective is to choose designs that minimize expected total distortion over a finite horizon, expected discounted distortion over an infinite horizon and average distortion per unit time over an infinite horizon. The above problem is a dynamic team with non-classical information structure. We develop a sequential decomposition for this problem. The main contribution of this paper is to provide a systematic methodology for determination of optimal joint source-channel encoding-decoding strategies for zero-delay or fixed finite-delay point-to-point communication with limited memory.**

I. Introduction

CONSIDER A FIRST ORDER Markov source that generates output $\{X_n\}_{n=1}^{\infty}$ belonging to a finite alphabet $\mathcal{X} \triangleq \{1, \dots, |\mathcal{X}|\}$ with a known transition matrix P. The probability distribution P_{X_0} of the initial state X_0 is known. The source outputs X_n are encoded by a transmitter with finite memory, the encoded symbols are transmitted over a discrete memoryless channel (DMC) and the received symbols are decoded by a receiver with finite memory. The distortion metric accepts a fixed delay of D units i.e., for any n, the receiver must generate an estimate of x_n at time $n + D$.

We model the transmitter and receiver as finite state machines (FSMs). The encoded symbol Z_n belongs to a discrete alphabet $\mathcal{Z} \triangleq \{1, \dots, |\mathcal{Z}|\}$ and is generated as follows

$$Z_n = f(X_n, S_{n-1}), \qquad S_n = h(X_n, S_{n-1}),$$

where $f(\cdot)$ is the encoding function, $h(\cdot)$ is the transmitter's memory update function and S_n represents the transmitter's memory contents at time n. The transmitter has finite memory, i.e., S_n takes values in a finite set $\mathcal{S} \triangleq \{1, \dots, |\mathcal{S}|\}$. The encoded symbol Z_n is transmitted over a DMC producing Y_n that belongs to $\mathcal{Y} \triangleq \{1, \dots, |\mathcal{Y}|\}$. We assume that the channel statistics are known, i.e.,

$$\Pr\left(y_n \,\middle|\, z^n, x^n, y^{n-1}\right) = \Pr\left(y_n \,\middle|\, z_n\right) = Q(y_n, z_n), \qquad (1)$$

where Q is a known transition matrix of the channel. The receiver has finite memory. The receiver's memory contents at time n is denoted by M_n which takes values in a finite set $\mathcal{M} \triangleq \{1, \dots, |\mathcal{M}|\}$. The receiver generates an estimate of the source $\widehat{X}_n$, belonging to $\mathcal{X}$, as follows:

$$\widehat{X}_n = g(Y_n, M_{n-1}), \qquad M_n = l(Y_n, M_{n-1}),$$

where $g(\cdot)$ is the decoding function, $l(\cdot)$ is the receiver's memory update function. A uniformly bounded distortion metric $\rho : \mathcal{X} \times \mathcal{X} \to [0, K], K < \infty$ is given and can accept a delay of D time units i.e. the distortion at time $n, n \geq D$ is given by $\rho(x_{n-D}, \hat{x}_n)$. Thus, we require the delay in reconstruction to be equal to D, where D is a pre-specified constant. This restriction is stronger than requiring the delay in reconstruction to be finite.

We shall refer to $|\mathcal{S}|$ as the "size of transmitter memory", to $|\mathcal{M}|$ as the "size of receiver memory" and to D as the "delay acceptable by the distortion measure" or simply as "decoding delay" or "delay". We shall use the terms "distortion" and "cost" interchangeably. The choice of $f(\cdot)$, $g(\cdot)$, $h(\cdot)$, $l(\cdot)$ is called a *design*. We quantify the performance of a design by the corresponding expected distortion per unit time, i.e.,

$$\mathcal{J}(f, g, h, l) \triangleq \lim_{N \to \infty} \mathbb{E}\left\{ \frac{1}{\widetilde{N}} \sum_{n=D+1}^{N} \rho(X_{n-D}, \widehat{X}_n) \,\middle|\, f, g, h, l \right\}.$$

where $\widetilde{N} = N - D$. The problem under consideration is as follows:

Problem 1: *Given a system $(\mathcal{X}, \mathcal{Z}, \mathcal{Y}, \mathcal{S}, \mathcal{M}, P_{X_0}, P, Q)$, a uniformly bounded distortion $\rho(\cdot)$, delay D and any $\varepsilon > 0$, choose a design (f^*, g^*, h^*, l^*) that performs ε close to the minimum distortion, i.e.,*

$$\begin{aligned}\mathcal{J}(f^*, g^*, h^*, l^*) - \varepsilon \leq \mathcal{J}^* &\triangleq \min_{f,g,h,l} \mathcal{J}(f, g, h, l) \\ &= \min_{f,g,h,l} \lim_{N \to \infty} \mathbb{E}\left\{ \frac{1}{\widetilde{N}} \sum_{n=D+1}^{N} \rho(X_{n-D}, \widehat{X}_n) \,\middle|\, f, g, h, l \right\}.\end{aligned}$$

where $\widetilde{N} = N - D$.

The salient features of this problem are: (i) finite size of transmitter and receiver memory; and (ii) fixed finite decoding delay. By sufficiently increasing the size of transmitter and receiver memory $|\mathcal{S}|, |\mathcal{M}|$ and delay D, results of rate-distortion theory become applicable [1, Theorem 2]. We are however, concerned with the case where $|\mathcal{S}|$, $|\mathcal{M}|$ and D are finite and small.

The motivation for studying this problem comes from problems related to resource allocation in networks with QoS requirements (e.g. end-to-end delay), distributed routing in wired and wireless networks, decentralized detection in sensor networks, and traffic flow control in transportation networks. Delay in information transmission/exchange is an important consideration for many such applications but is not taken into consideration in classical information theory. Further, devices are getting smaller in size, but the size of batteries is not reducing at the same rate. Small, high density energy storage batteries are not available for small devices. This restricts the power consumption and consequently the data processing capabilities of such devices. Thus, even if such devices have large memory, not all of it can be processed. Moreover, for certain applications, having a large memory results in an unacceptably high cost of the device. Hence, we want to obtain a systematic methodology for finding optimal or nearly optimal designs and to investigate good design heuristics for delay and memory limited communication systems.

The problem with zero (and finite) delay coding, infinite size of transmitter and receiver memory and noiseless channels was

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considered in [2], [3]; a similar setting for noisy channels was considered in [4]. The problem with zero (and finite) delay coding, infinite size of transmitter memory, finite size of receiver memory and noiseless channel was considered in [5]. The same setting with noisy channels was considered in [6], [7], [8]. The case of zero (and finite) delay coding with finite size of transmitter and receiver memory for noiseless channels was considered in [9], [10]. The results of [9], [10] for the case of Markov sources were inconclusive and the authors concluded by "The subject of finite-state machines driven by Markov inputs warrants further investigation".

We use stochastic optimization theory to investigate the zero (or fixed finite) delay encoding, decoding and memory update problem with finite memory transmitters and receivers and a noisy channel. Problem 1 is difficult to analyze, because for each design we have to evaluate its asymptotic performance. We circumvent this difficulty as follows. We start with a simpler problem, namely, a zero-delay finite horizon problem (Section II) where the functions $f(\cdot), g(\cdot), h(\cdot), l(\cdot)$ are allowed to be time varying. We develop a methodology to solve this problem. We extend the solution methodology to an infinite horizon, zero-delay expected discounted distortion problem in Section III. We consider the zero-delay infinite horizon average distortion per unit time problem in Section IV; we show that for a discount factor close to 1, the solution of the problem of Section III is also a solution of the problem of Section IV. Thus, we have a methodology for solving Problem 1 with delay $D = 0$. In Section V we explain how to transform the fixed finite delay problem to a zero-delay problem. Consequently, we have a solution methodology for Problem 1. There exist polynomial complexity algorithms to solve the problem of Section III. Thus, these algorithms can be used to obtain a solution of Problem 1.

II. The Finite Horizon Problem

In this section we consider a zero-delay finite horizon version of Problem 1, i.e., $D = 0$. The encoding, decoding, transmitter and receiver memory update functions are allowed to be time varying.

A. Problem Formulation

Consider a first order Markov source that generates a random sequence $X_1, \dots, X_N$, for each $n = 1, \dots, N$, $X_n \in \mathcal{X}$. The transition matrix P is known and $P_{ij} \triangleq \Pr(X_{n+1} = j \mid X_n = i), \forall i, j \in \mathcal{X}$. The probability distribution of the initial state X_0, denoted by P_{X_0}, is assumed to be known. The encoder is a FSM which generates a sequence $Z_1, \dots, Z_N$, $Z_n \in \mathcal{Z}$ as follows:

$$Z_n = f_n(X_n, S_{n-1}), \tag{2}$$
$$S_n = h_n(X_n, S_{n-1}), \tag{3}$$

where $S_n \in \mathcal{S}$ is the transmitter's memory content at time n. S_0 is arbitrarily initialized to 1, f_n belongs to $\mathscr{F}$, the family of functions from $\mathcal{X} \times \mathcal{S}$ to $\mathcal{Z}$ and h_n belongs to $\mathscr{H}$, the family of functions from $\mathcal{X} \times \mathcal{S}$ to $\mathcal{S}$. The encoded sequence is transmitted over a DMC, given by (1), producing $Y_1, \dots, Y_N$. The decoder is a FSM which generates a sequence $\widehat{X}_1, \dots, \widehat{X}_N$, $\widehat{X}_n \in \mathcal{X}$ as follows:

$$\widehat{X}_n = g_n(Y_n, M_{n-1}), \tag{4}$$
$$M_n = l_n(Y_n, M_{n-1}), \tag{5}$$

where $M_n \in \mathcal{M}$ is the receiver's memory content at time n and M_0 is arbitrarily initialized to 1, g_n belongs to $\mathscr{G}$, the family of function from $\mathcal{Y} \times \mathcal{M}$ to $\mathcal{X}$ and l_n belongs to $\mathscr{L}$, the family of function from $\mathcal{Y} \times \mathcal{M}$ to $\mathcal{M}$. A bounded distortion metric $\rho : \mathcal{X} \times \mathcal{X} \to [0, K], K < \infty$ is given and the acceptable decoding delay $D = 0$. The choice of $f \triangleq (f_1, \dots, f_N)$, $g \triangleq (g_1, \dots, g_N)$, $h \triangleq (h_1, \dots, h_N)$ and $l \triangleq (l_1, \dots, l_N)$ is called a *design*. Denote by $\Gamma^N \triangleq (\mathscr{F} \times \mathscr{G} \times \mathscr{H} \times \mathscr{L})^N$ the family of all designs. The performance of any design is given by an expected total distortion,

$$\mathcal{J}_N(f, g, h, l) \triangleq \mathbb{E}\left\{ \sum_{n=1}^{N} \rho(X_n, \widehat{X}_n) \,\middle|\, f, g, h, l \right\}. \tag{6}$$

The optimization problem that we consider is as follows:

Problem 2: *Given a system* $(\mathcal{X}, \mathcal{Z}, \mathcal{Y}, \mathcal{S}, \mathcal{M}, P_{X_0}, P, Q, N)$*, a bounded distortion* $\rho(\cdot)$ *and delay* $D = 0$*, choose a design* (f^*, g^*, h^*, l^*) *that is optimal with respect to the performance criterion given by* (6)*, i.e.,*

$$\mathcal{J}_N(f^*, g^*, h^*, l^*) = \mathcal{J}_N^* \triangleq \min_{(f,g,h,l) \in \Gamma^N} \mathcal{J}_N(f, g, h, l). \tag{7}$$

B. Joint Optimization

Problem 2 is a *dynamic team* with non-classical information pattern [11]. In this section we find information states sufficient for performance evaluation, thereby obtaining a sequential decomposition of the problem.

Definition 1: *Let* Π *be the space of probability measures on* $\mathcal{X} \times \mathcal{Y} \times \mathcal{S} \times \mathcal{M}$ *and* Ψ *be the space of probability measures on* $\mathcal{X} \times \mathcal{S} \times \mathcal{M}$*. Define the following for all* $n = 1, \dots, N$*,*

$$\pi^0_{n-1} = \Pr(X_{n-1}, S_{n-1}, M_{n-1}), \tag{8}$$
$$\pi^1_n = \Pr(X_n, Y_n, S_{n-1}, M_{n-1}), \tag{9}$$
$$\pi^2_n = \Pr(X_n, S_{n-1}, M_n). \tag{10}$$

Observe that $\pi^1_n \in \Pi$, $\pi^0_n, \pi^2_n \in \Psi$ and π^0_0 is known.

Lemma 1: *For all* $n = 1, \dots, N$*,*

(i) there exist linear transformations $T^0_{n-1}(f_n), T^1_n(l_n), T^2_n(h_n)$ *such that*

$$\pi^1_n = T^0_{n-1}(f_n)\pi^0_{n-1}. \tag{11}$$
$$\pi^2_n = T^1_n(l_n)\pi^1_n. \tag{12}$$
$$\pi^0_n = T^2_n(h_n)\pi^2_n. \tag{13}$$

(ii) for any choice of f, g, h, l*, the expected instantaneous cost can be expressed as*

$$\mathbb{E}\left\{ \rho(X_n, \widehat{X}_n) \,\middle|\, f^n, g^n, h^n, l^n \right\} = \tilde{\rho}(\pi^1_n, g_n). \tag{14}$$

where $\alpha^n = (\alpha_1, \dots, \alpha_n), \alpha = f, g, h, l$ *and* $\tilde{\rho}(\cdot)$ *is a deterministic function.*

Proof:

(i) Consider a component of π^1_n,

$$\pi^1_n(x, y, s, m) = \sum_{x_{n-1} \in \mathcal{X}} \pi^0_{n-1}(x_{n-1}, s, m) P_{x, x_{n-1}} \times Q_{y, f_n(x,s)} \triangleq \big(T^0_{n-1}(f_n)\pi^0_{n-1}\big)(x, y, s, m). \tag{15}$$

(ii) Consider a component of π^2_n,

$$\pi^2_n(x, s, m) = \sum_{m_{n-1} \in \mathcal{M}} \sum_{y_n \in \mathcal{Y}} \pi^1_n(x, y_n, s, m_{n-1}) \times \mathbb{1}\left[m = l_n(y_n, m_{n-1}) \right] \triangleq \big(T^1_n(l_n)\pi^1_n\big)(x, s, m), \tag{16}$$

where $\mathbb{1}[\cdot]$ is the indicator function.

(iii) Consider a component of π_n^0,

$$\pi_n^0(x,s,m) = \sum_{s_{n-1}\in\mathcal{S}} \pi_n^2(x,s_{n-1},m)\,\mathbb{1}[s = h_n(x,s_{n-1})] \triangleq \big(T_n^2(h_n)\pi_n^2\big)(x,s,m). \quad (17)$$

(iv) Consider the expected instantaneous cost,

$$\mathbb{E}\left\{ \rho(X_n,\widehat{X}_n) \,\middle|\, f^n,g^n,h^n,l^n \right\} = \sum_{\substack{x_n,y_n,m_{n-1}\in\\ \mathcal{X}\times\mathcal{Y}\times\mathcal{M}}} \rho\big(x_n, g_n(y_n,m_{n-1})\big) \times \sum_{s_{n-1}\in\mathcal{S}} \pi_n^1(x_n,y_n,s_{n-1},m_{n-1}) \triangleq \tilde{\rho}(\pi_n^1,g_n). \quad (18)$$

∎

The choice of f^n,g^n,h^n,l^n makes the variable $\widehat{X}_n$ a random variable with well-defined distribution. Thus, the performance measure of (6) can be rewritten as

$$\mathbb{E}\left\{ \sum_{n=1}^N \rho(X_n,\widehat{X}_n) \,\middle|\, f,g,h,l \right\} = \sum_{n=1}^N \mathbb{E}\left\{ \rho(X_n,\widehat{X}_n) \,\middle|\, f^n,g^n,h^n,l^n \right\} = \sum_{n=1}^N \tilde{\rho}(\pi_n^1,g_n) \quad (19)$$

where the sequence $\pi_1^1,\dots,\pi_N^1$ depends on the choice of f,h,l. Hence, Lemma 1 implies that Problem 2 is equivalent to the following deterministic optimization problem:

Problem 3: *Consider a discrete time, finite horizon, system that evolves as follows*

$$\pi_n^1 = T^0(f_n)\pi_{n-1}^0, \qquad n = 1,\dots,N, \quad (20)$$
$$\pi_n^2 = T^1(l_n)\pi_n^1, \qquad n = 1,\dots,N, \quad (21)$$
$$\pi_n^0 = T^2(h_n)\pi_n^2, \qquad n = 1,\dots,N, \quad (22)$$

where f_n,l_n,h_n are functions belonging to $\mathscr{F},\mathscr{L},\mathscr{H}$, respectively, and $T^i(\cdot), i = 0,1,2$ are known linear transforms. The initial state π_0^0 is known. When the system is in state π_n^1, for any choice of function g_n belonging to $\mathscr{G}$ an instantaneous cost $\tilde{\rho}(\pi_n^1,g_n)$ is incurred.

The objective is to choose function $f \triangleq (f_1,\dots,f_N)$, $g \triangleq (g_1,\dots,g_N)$, $h \triangleq (h_1,\dots,h_N)$, $l \triangleq (l_1,\dots,l_N)$ so as to minimize the total cost given by

$$\mathcal{J}_N(f,g,h,l) = \sum_{n=1}^N \tilde{\rho}(\pi_n^1,g_n). \quad (23)$$

This is a classical deterministic control problem and an optimal design is given by the following.

Theorem 1: *An optimal design for Problem 3, and consequently, Problem 2 is given by the solution of the following nested optimality gathers:*

$$V_N^2(\pi_N^2) \triangleq 0, \quad (24)$$

and for $n = 1,\dots,N$

$$V_{n-1}^0(\pi_{n-1}^0) = \min_{f_n\in\mathscr{F}} \left\{ V_n^1\big(T^0(f_n)\pi_{n-1}^0\big) \right\}, \quad (25)$$
$$V_n^1(\pi_n^1) = \min_{l_n\in\mathscr{L}} \left\{ \overline{V}_n(\pi_n^1) + V_n^2\big(T^1(l_n)\pi_n^1\big) \right\}, \quad (26)$$
$$\overline{V}_n(\pi_n^1) = \min_{g_n\in\mathscr{G}} \left\{ \tilde{\rho}(\pi_n^1,g_n) \right\}, \quad (27)$$
$$V_n^2(\pi_n^2) = \min_{h_n\in\mathscr{H}} \left\{ V_n^0\big(T^2(h_n)\pi_n^2\big) \right\}, \quad (28)$$

The arg min *at each step determines the corresponding optimal design rule. Furthermore, the optimal performance is given by*

$$\mathcal{J}_N^* = V_0^0(\pi_0^0) \quad (29)$$

Proof: This is a standard result, see [12, Chapter 2]. ∎

III. The Expected Discounted Cost Problem

Consider the following extension of the model of Section II-A to an infinite horizon ($N \to \infty$). The performance of any design is evaluated by the expected discounted distortion under that design given by

$$\mathcal{J}^\beta(f,g,h,l) \triangleq \mathbb{E}\left\{ \sum_{n=1}^\infty \beta^{n-1}\rho(X_n,\widehat{X}_n) \,\middle|\, f,g,h,l \right\}, \quad (30)$$

where $0 < \beta < 1$ is called the discount factor. Lemma 1 implies that this problem is equivalent to the following deterministic problem.

Problem 4: *Consider a discrete time, infinite horizon, system that evolves as follows for all n:*

$$\pi_n^1 = T^0(f_n)\pi_{n-1}^0, \quad (31)$$
$$\pi_n^2 = T^1(l_n)\pi_n^1, \quad (32)$$
$$\pi_n^0 = T^2(h_n)\pi_n^2, \quad (33)$$

where f_n,l_n,h_n are functions belonging to $\mathscr{F},\mathscr{L},\mathscr{H}$ respectively and $T^i(\cdot), i = 0,1,2$ are known transforms. The initial state π_0^0 is known. When the system is in state π_n^1, for any choice of function g_n belonging to $\mathscr{G}$ an instantaneous cost $\tilde{\rho}(\pi_n^1,g_n)$ is incurred.

The objective is to choose functions $f \triangleq (f_1,f_2,\dots)$, $g \triangleq (g_1,g_2,\dots,)$, $h \triangleq (h_1,h_2,\dots,)$, $l \triangleq (l_1,l_2,\dots,)$ so as to minimize the discounted cost over infinite horizon given by

$$\mathcal{J}^\beta(f,g,h,l) = \sum_{n=1}^\infty \beta^{n-1}\tilde{\rho}(\pi_n^1,g_n). \quad (34)$$

Let $\gamma_n = (f_n,g_n,h_n,l_n)$ Define a transformation $\widehat{T}(\gamma) \triangleq T^0(f_n)\circ T^1(l_n)\circ T^2(h_n)$. Then (31)–(33) can be written as

$$\pi_n^0 = \widehat{T}(\gamma_n)\pi_{n-1}^0, \quad (35)$$

and we can define $\hat{\rho}(\pi_{n-1}^0,\gamma_n)$ by

$$\hat{\rho}(\pi_{n-1}^0,\gamma_n) \triangleq \tilde{\rho}\big(T^0(f_n)\pi_{n-1}^0,g_n\big) = \tilde{\rho}(\pi_n^1,g_n). \quad (36)$$

Then (34) can be rewritten as

$$\mathcal{J}^\beta(\gamma) = \sum_{n=1}^\infty \beta^{n-1}\hat{\rho}(\pi_{n-1}^0,\gamma_n), \quad (37)$$

where π_0^0 is known.

The system of gathers (35), (36), (37) is the classical expected discounted cost problem whose solution is given by the following.

Theorem 2: *Consider the optimization problem described by* (35), (36), (37). *An optimal solution is given by a stationary policy $\gamma^\infty \triangleq (\gamma,\gamma,\dots,)$, i.e.*

$$V(\pi_0^0) = \mathcal{J}^\beta(\gamma^\infty) = \min_{\gamma'\in\Gamma^\infty} \mathcal{J}^\beta(\gamma'), \quad (38)$$

where V is the unique uniformly bounded *fixed point of*

$$V(\pi) = \min_{\gamma\in\Gamma} \left\{ \hat{\rho}(\pi,\gamma) + \beta V\big(\widehat{T}(\gamma)\pi\big) \right\}, \quad \forall \pi \in \Pi. \quad (39)$$

Proof: This is a standard result. See [12]. ∎

IV. The Average Cost per Unit Time Problem

Consider the following extension of the model of Section II-A to an infinite horizon ($N \to \infty$). The performance of any design is evaluated by the expected average distortion per unit time under that design which is given by

$$\overline{\mathcal{J}}(f,g,h,l) \triangleq \limsup_{N\to\infty} \frac{1}{N}\mathbb{E}\left\{ \sum_{n=1}^{N} \rho(X_n,\widehat{X}_n) \;\middle|\; f,g,h,l \right\}. \tag{40}$$

Lemma 1 implies that this problem is equivalent to a deterministic problem described in Problem 4 with an optimization criterion given by

$$\overline{\mathcal{J}}(f,g,h,l) = \limsup_{N\to\infty} \frac{1}{N}\mathbb{E}\left\{ \sum_{n=1}^{N} \rho(X_n,\widehat{X}_n) \right\}. \tag{41}$$

Using the transformations $\widehat{T}(\gamma_n)$ and $\hat{\rho}(\pi^0_{n-1},\gamma_n)$ defined by (35) and (36), we can rewrite (41) as

$$\overline{\mathcal{J}}(\gamma) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \hat{\rho}(\pi^0_{n-1},\gamma_n). \tag{42}$$

The system of gathers (35), (36), (42) is the classical average cost per unit time problem, whose solution is given by the following:

Theorem 3: *For the optimization problem described by* (35)*,* (36)*,* (42)*, and assume*

(A1) *for some $\varepsilon > 0$ there exist bounded measurable functions $v(\cdot)$ and $r(\cdot)$ and design $\gamma_0 \in \Gamma$ such that for all $\pi \in \Pi$,*

$$v(\pi) = \min_{\gamma\in\Gamma}\left\{ v\big(\widehat{T}(\gamma)\pi\big)\right\} = v\big(\widehat{T}(\gamma_0)\pi\big), \tag{43}$$

$$\begin{aligned}\min_{\gamma\in\Gamma}\left\{ \hat{\rho}(\pi,\gamma) + r\big(\widehat{T}(\gamma)\pi\big)\right\} &\le v(\pi) + r(\pi) \\ &\le \hat{\rho}(\pi,\gamma_0) + r\big(\widehat{T}(\gamma_0)\pi\big) + \varepsilon.\end{aligned} \tag{44}$$

Then there exists an optimal stationary design $\gamma_0^\infty = (\gamma_0,\gamma_0,\dots)$ such that: for any horizon N and any design γ' for that horizon

$$\mathcal{J}_N(\gamma_0^N) \le r(\pi_0^0) + N v(\pi_0^0) \le \mathcal{J}_N(\gamma') + N\varepsilon, \tag{45}$$

where $\gamma_0^N \triangleq (\gamma_0,\dots,\gamma_0)$ (N terms). Furthermore, under (A1), (45) *is equivalent to*

$$\overline{\mathcal{J}}(\gamma_0^\infty) = v(\pi_0^0) \le \underline{\mathcal{J}}(\gamma') + \varepsilon, \tag{46}$$

where γ' is any policy for the infinite horizon and

$$\underline{\mathcal{J}}(\gamma') = \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \hat{\rho}(\pi^0_{n-1},\gamma'_n). \tag{47}$$

Proof: (43) and (44) are the canonical form of the system (35), (36), (42) [13, Chapter 7]. For this canonical system, if a measurable bounded solution exists, it is optimal in the sense of (45) and (46) [13, Chapter 7]. ■

V. The Fixed Finite Delay Problem

In this section we consider the case when the acceptable delay D is not zero. We first consider finite horizon systems and later extend the methodology for infinite horizon systems.

A. Fixed Finite Delay Finite Horizon Problem

The basic model of Section II can be modified as follows when a delay $D > 0$ in included in the definition of distortion. The first change is that the variables $\widehat{X}_1,\dots,\widehat{X}_D$ are simply not generated, the receiver spends its first D periods just accumulating the observations $Y_1,\dots,Y_D$ and updating its memory accordingly.

The second change is that for $n > D$, distortion is measured by a function $\rho(X_{n-D},\widehat{X}_n)$. The objective is to choose $f \triangleq (f_1,\dots,f_N)$, $g \triangleq (g_1,\dots,g_N)$, $h \triangleq (h_1,\dots,h_N)$, $l \triangleq (l_1\dots,l_N)$ to minimize expected distortion given by

$$\mathcal{J}_N^D(f,g,h,l) \triangleq \mathbb{E}\left\{ \sum_{n=D+1}^{N} \rho(X_{n-D},\widehat{X}_n) \;\middle|\; f,g,h,l \right\}. \tag{48}$$

We will follow the *sliding window* repackaging of the source as in [5], [6], [7]. Define the process $\{\overline{X}_1,\dots,\overline{X}_N\}$ where

$$\overline{X}_n \triangleq \begin{cases} \big(X_1,\dots,X_n\big), & n \le D, \\ \big(X_{n-D},X_{n-D+1},\dots,X_n\big), & n > D. \end{cases} \tag{49}$$

Definition 2: *Define for all n*

$$\pi^0_{n-1} = \Pr\left(\overline{X}_{n-1},S_{n-1},M_{n-1}\right), \tag{50}$$

$$\pi^1_n = \Pr\left(\overline{X}_n,Y_n,S_{n-1},M_{n-1}\right), \tag{51}$$

$$\pi^2_n = \Pr\left(\overline{X}_n,,S_{n-1},M_n\right). \tag{52}$$

It can be shown that the information states of Definition 2 satisfy the properties of Lemma 1[1]. Thus, we can formulate a deterministic problem in the same way as in Problem 3 which can be solved as follows.

Theorem 4: *An optimal design with respect to the optimization criterion given by* (48) *is obtained by the solution of the nested optimality gathers* (24)–(28) *with* (27) *replaced by*

$$\overline{V}_n(\pi^1_n) = \begin{cases} \min\limits_{g_n\in\mathscr{G}} \left\{\tilde{\rho}(\pi^1_n,g_n)\right\}, & n = D+1,\dots,N, \\ 0, & n = 1,\dots,D. \end{cases} \tag{53}$$

B. Fixed Finite Delay Infinite Horizon Problem

In the infinite horizon problems, we again use the sliding window repackaging of the source. The probability measures π^i_n, $i = 0,1,2$ given by Definition 2 are still information states for the problem and satisfy Lemma 1. However, the system is no longer time-homogeneous as the instantaneous cost is zero for the first D time steps and given by $\tilde{\rho}(\cdot)$ after that. We can break the problem into two phases (i) initialization phase for $n = 1,2,\dots,D$, and (ii) sliding window phase for $n = D+1, D+2, \dots$. For the sliding window phase, we have a time-homogeneous infinite horizon problem that can be solved using the optimality gathers of Theorem 2 for the expected discounted cost problem and of Theorem 3 for the average cost per unit time problem. The solution will give the *cost to go* from time $D+1$ onward for all $\pi_D \in \Pi$ and an optimal stationary policy from time $D+1$ onward. To find an optimal policy in the initialization phase, we can view the problem as a finite horizon system with horizon D with zero instantaneous cost and stopping cost given by cost to go functions obtained from the solution of the sliding window phase.

[1]The instantaneous cost for first D time instances is 0.

A special case: In the average cost per unit time problem, if the differential cost function $v(\pi)$ is constant equal to v for all $\pi \in \Pi$, then we can use any policy (in particular, the stationary policy that is also optimal for the sliding window phase) in the initialization phase. There is no instantaneous cost and the cost to go from any stopping state is the same. However, if $v(\pi)$ is not a constant, then we have to choose a policy for the initialization phase so as to drive the system to the "best" π_D. Similar argument holds for the expected discounted cost problem.

VI. Relation between the two infinite horizon problems

The average cost per unit time problem is related to the expected discounted cost problem via a Tauberian theorem [14, Theorem A.4.2].

Theorem 5: *Let u_n be a sequence of non-negative terms. Let $U(\beta) \triangleq \sum_{n=1}^{\infty} \beta^{n-1} u_n$, for $0 < \beta < 1$, and $S_n \triangleq \sum_{k=1}^{n} u_k$ for $n \geq 1$. Then*

$$\liminf_{n\to\infty} \frac{S_n}{n} \leq \liminf_{\beta\to 1^-} (1-\beta)U(\beta) \leq \leq \limsup_{\beta\to 1^-} (1-\beta)U(\beta) \leq \limsup_{n\to\infty} \frac{S_n}{n}. \quad (54)$$

Further, the following statements are equivalent

(i) All terms in (54) *are equal and finite.*
(ii) $\lim_{n\to\infty} S_n/n$ *exists and is finite.*
(iii) $\lim_{\beta\to 1}(1-\beta)U(\beta)$ *exists and is finite.*

This theorem implies that if condition $(*)$ given by

$(*)$ $\lim_{\beta\to 1}(1-\beta)J^{\beta,*}$ exists and is finite.

holds, an optimal solution of expected discounted cost problem with discount factor β close to 1 is also an optimal solution for the average cost per unit time problem. Conditions sufficient for $(*)$ to hold have been studied in the literature [15]. Consider the expected discounted cost problem with discount factor β close to 1. A unique uniformly bounded solution for this problem is given by the unique fixed point of the contraction map given by (39). There exist polynomial complexity algorithms to come arbitrarily close (within an ε) to this fixed point (see [16]) and obtain the corresponding ε-optimal stationary policy. If $(*)$ holds, this ε-optimal policy for the expected discounted cost problem is also ε-optimal for the average cost per unit time problem. This provides an algorithm to obtain a solution of Problem 1 under conditions when $(*)$ holds. The conditions on $\rho(\cdot)$ that will guarantee that $(*)$ holds need further investigation.

VII. Some Special Cases

A. A fixed delay channel coding problem

Consider the channel coding problem of choosing an optimal finite memory receiver for a given convolution encoder for i.i.d. source outputs and given fixed delay D. There are various techniques for decoding convolutional codes with fixed delay. Typically, a sliding window of observations at the receiver are assumed and most researchers have focused on obtaining computationally efficient algorithms to determine the MAP bit decoding rule. However, as far as the authors are aware, the problem of optimally storing the observations has not been considered. If the receiver has a memory $|\mathcal{M}| = k|\mathcal{Y}|$, is it optimal to store the previous k channel observations? Can the receiver somehow "compress" all the past observations in $k|\mathcal{Y}|$ and get better performance? If so, how can such optimal "compression" functions be found. This problem fits naturally into the framework provided in this paper.

B. A finite delay source coding problem

Consider the source coding problem of choosing optimal finite-delay limited memory encoding-decoding schemes for noiseless channels for Markov sources. This problem fits naturally in the framework provided in this paper and is a generalization of the results in [5], [10]. Source coding theorists are also interested in techniques for finite-delay limited memory encoding-decoding schemes for individual sequences. The decision theoretic methodology provided herein can be easily extended to non-stochastic min-max problems and thus used to study source coding of individual sequences.

VIII. Conclusion

We have presented a methodology for fixed delay optimal joint source-channel coding or finite memory systems. We have also identified an algorithm to obtain optimal designs.

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