Team optimal decentralized state estimation

Aditya Mahajan and Mohammad Afshari
McGill University

IEEE Conference on Decision and Control
19 December 2018

Separation of Estimation and Control for Discrete Time Systems

HANS S. WITSENHAUSEN, MEMBER, IEEE

Invited Paper

Let's revisit separation of estimation and control in centralized systems

Separation in estimation and control

STANDARD LQG MODEL

$$x(t+1) = Ax(t) + Bu(t) + w(t),$$

$$y(t) = Cx(t) + v(t).$$

Choose $u(t) = g_t(y(1:t), u(1:t-1))$ to

$$\min \mathbb{E}\left[\sum_{t=1}^{T} \left[x(t)^\top Q x(t) + u(t)^\top R u(t)\right]\right]$$

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Total cost can be written as

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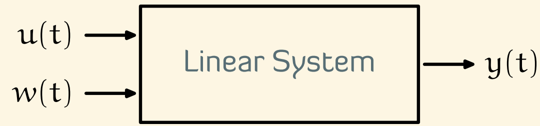

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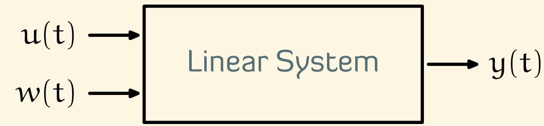

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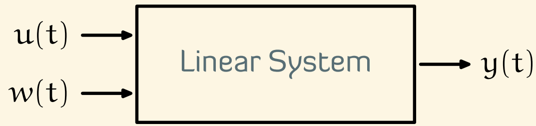

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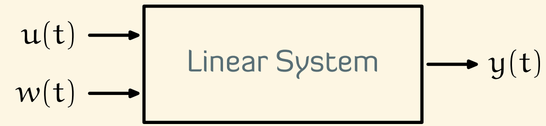

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STATIC REDUCTION

$$\sigma(y(1:t), u(1:t-1)) = \sigma(\tilde{y}(1:t-1)).$$

Thus, wlog, consider $\hat{z}(t) = \tilde{g}_t(\tilde{y}(1:t))$.

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Thus, $\hat{z}(t) = -L\ \mathbb{E}[\tilde{x}(t) \,|\, \tilde{y}(1:t)]$

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Motivation for current work

Separation centralized stochastic control, the optimal control action depends on the solution of an estimation problem:

$$\mathbb{E}\Big[\sum_{t=1}^{T}(L(t)\tilde{x}(t)+\hat{z}(t))^{\top}S(t)(L(t)\tilde{x}(t)+\hat{z}(t))\Big]$$

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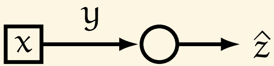
There is a long history of **duality** between estimation and control.

Decentralized control is interesting. Ergo, decentralized estimation is interesting.

Decentralized estimation is interesting in it's own right in certain applications.

DECENTRALIZED state estimation is **fundamentally different** from CENTRALIZED state estimation.

Centralized estimation



$$x \sim \mathcal{N}(0, \Sigma_x),$$

$$y = Cx + v, \qquad v \sim \mathcal{N}(0, R).$$

Choose $\hat{z} = g(y)$ to

minimize $\mathbb{E}\big[(Lx - \hat{z})^\top S (Lx - \hat{z})\big]$.

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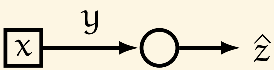

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where $K = \Sigma_x C^\top (C \Sigma_x C^\top + R)^{-1}$.

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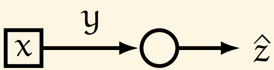

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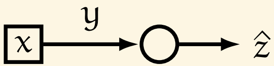
Centralized estimation vs decentralized estimation



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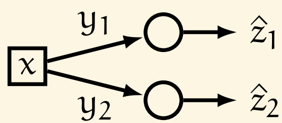

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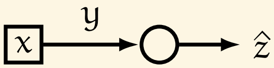

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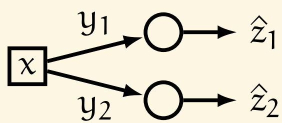

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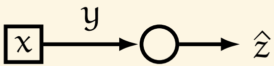

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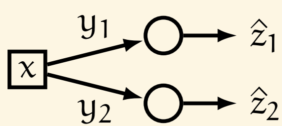

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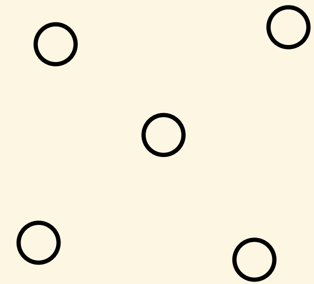

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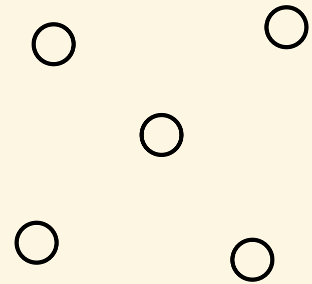
The system consists of n agents.

$$y_i(t) = C_i x(t) + v_i(t), \quad v_i(t) \sim \mathcal{N}(0, R_i).$$



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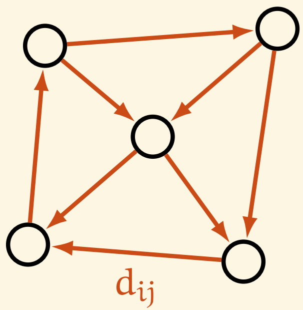
INFO STRUCTURE

- Agents communicate over a strongly connected weighted directed graph.
- Edge weight d_{ij} corresponds to link delay.

$$I_i(t) = \{y_i(1:t)\} \cup \Big(\bigcup_{j \in N_i^-} I_j(t - d_{ji}) \Big)$$



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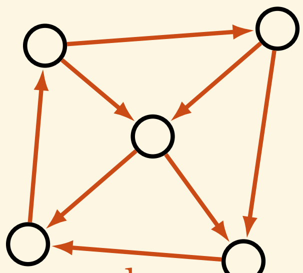

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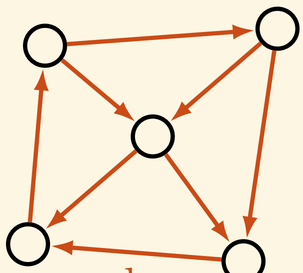

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NEIGHBORHOOD SHARING

$$I_i(t) = \bigcup_{k=0}^{d^*} \bigcup_{j \in N_i^k} \{y_j(1:t-k)\}.$$

System model (continued)

ESTIMATES Each agent generates an estimate

$$\hat{z}_i(t) = g_{i,t}(I_i(t))$$

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PER-STEP ERROR

Let $\hat{z}(t) = \text{vec}(\hat{z}_1(t), \dots, \hat{z}_n(t))$. Then,

$$c(x(t), \hat{z}(t)) = \big(Lx(t) - \hat{z}(t)\big)^\top S \big(Lx(t) - \hat{z}(t)\big).$$

where

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$

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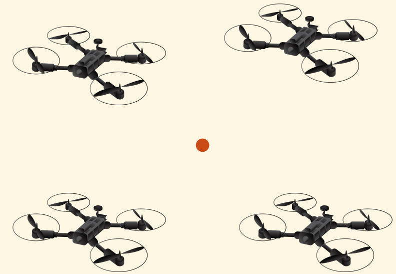

$$\sum_{i \in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \lambda \|\bar{x}(t) - \bar{z}(t)\|^2$$

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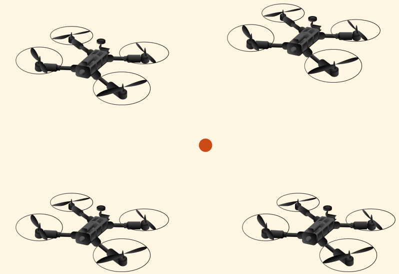
$$c(x(t), \hat{z}(t)) = \big(Lx(t) - \hat{z}(t)\big)^\top S\big(Lx(t) - \hat{z}(t)\big).$$

where

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$



$$\sum_{i \in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \lambda \|\bar{x}(t) - \bar{z}(t)\|^2$$

$$\sum_{i \in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \sum_{i=1}^{n-1} \lambda \|d_i(t) - \hat{d}_i(t)\|^2$$

System model (continued)

ESTIMATES Each agent generates an estimate

$$\hat{z}_i(t) = g_{i,t}(I_i(t))$$

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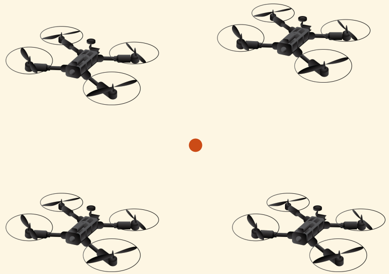
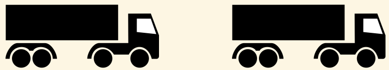
OBJECTIVE Choose a team estimation problem g to

$$\min \mathbb{E}^g \left[\sum_{t=1}^{T} c(x(t), \hat{z}(t)) \right]$$



$$\sum_{i \in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \lambda \|\bar{x}(t) - \bar{z}(t)\|^2$$



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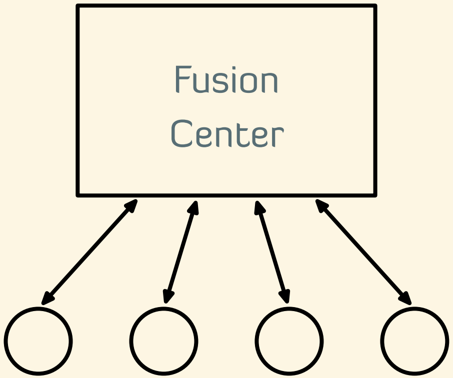
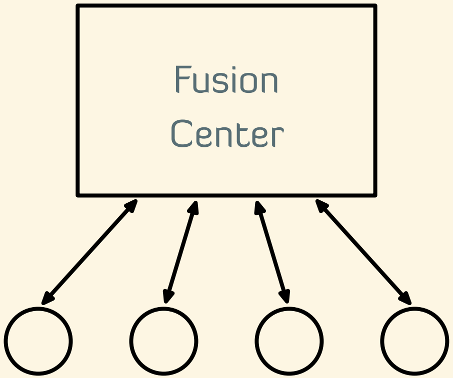
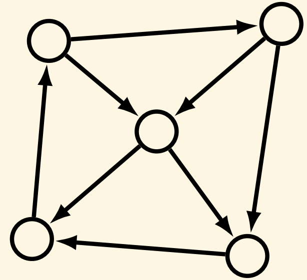
Literature Overview



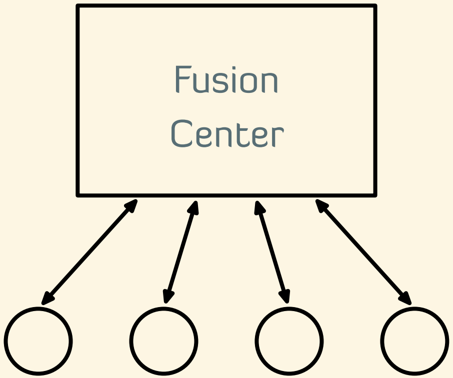
Literature Overview

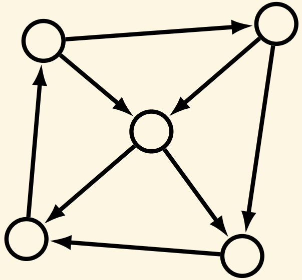




Conensus based methods

Literature Overview





Conensus based methods

TEAM OPTIMAL DECENTRALIZED ESTIMATION

- Barta, PhD Thesis (1978)
- Castanon, LIDS Tech Report (1981)
- Andersland and Teneketzis, JOTA (1996)

Solution approach: Witsenhausen's intrinsic model

1.1	1.2	$\cdots$	1.t	$\cdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n.1	n.2	$\cdots$	n.t	$\cdots$

Solution approach: Witsenhausen's intrinsic model

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Decentralized estimation is a STATIC TEAM problem.

Solution approach: Witsenhausen's intrinsic model

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Stand-alone optimization problem

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$$\begin{array}{ccccc} 1.1 & 1.2 & \cdots & 1.t & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ n.1 & n.2 & \cdots & n.t & \cdots \end{array}$$

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Stand-alone optimization problem

Decentralized estimation is a STATIC TEAM problem.

Instead of solving $\min \mathbb{E}\Big[\sum_{t=1}^{T} c(x(t), \hat{z}(t))\Big]$

we solve $\min \mathbb{E}[c(x(t), \hat{z}(t))]$ at each t

Solution approach: Witsenhausen's intrinsic model

$$\begin{matrix} 1.1 & 1.2 & \cdots & 1.t & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ n.1 & n.2 & \cdots & n.t & \cdots \end{matrix}$$

Stand-alone optimization problem

Decentralized estimation is a STATIC TEAM problem.

Instead of solving $\min \mathbb{E}\Big[\sum_{t=1}^{T} c(x(t), \hat{z}(t))\Big]$

we solve $\min \mathbb{E}[c(x(t), \hat{z}(t))]$ at each t

Decentralized estimation is a SEQUENCE of static team problems.

A naive application of
Radnar's result does not work.

Directly applying Radnar's result

OPTIMAL STRATEGY: $\hat{z}_i(t) = F_i(t) I_i(t)$

$\{F_i(t)\}_{i \in N}$ given by the solution of a system of matrix equations.

Directly applying Radnar's result

OPTIMAL STRATEGY: $\hat{z}_i(t) = F_i(t) I_i(t)$

$\{F_i(t)\}_{i \in N}$ given by the solution of a system of matrix equations.

$I_i(t)$ increases with time; so does the dimension of $F_i(t)$.

Complexity of finding the optimal solution increases with time.

Alternative idea: Common information approach

Common Information $$I^{\text{com}}(t) = \bigcap_{i \in N} I_i(t),$$

Local Information $$I_i^{\text{loc}}(t) = I_i(t) \setminus I^{\text{com}}(t),$$

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State estimate $\hat{x}^{\text{com}}(t) = \mathbb{E}[x(t) \mid I^{\text{com}}(t)].$

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Let $\hat{\Sigma}_{ij}(t) = \mathrm{cov}(\tilde{I}_i(t), \tilde{I}_j(t)).$

and $\hat{\Theta}_i(t) = \mathrm{cov}(x(t), \tilde{I}_i(t))$

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STRUCTURE OF OPTIMAL ESTIMATORS

$$\hat{z}_i(t) = L_i\, \hat{x}^{\text{com}}(t) + F_i(t)\, \tilde{I}_i^{\text{loc}}(t)$$

1st term: Common info based estimate

2nd term: Local innovation based correction (depends on weight matrix)

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COMPUTING OPTIMAL GAINS

System of matrix equations: for all $i \in N$,

$$\sum_{j \in N} \left[S_{ij} F_j(t) \hat{\Sigma}_{ji}(t) - S_{ij} L_j \hat{\Theta}_j(t) \right] = 0.$$

Alternative idea: Common information approach

PROOF IDEA

- Same as Radnar.
- Show that the proposed strategy is PBPO
- For convex static teams:

  PBPO $\Longrightarrow$ global optimal.

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VECTORIZED SOLUTION

Equivalen to

$$F(t) = \Gamma(t)^{-1}\eta(t)$$

where $F(t) = \text{vec}(F_1(t), \dots, F_n(t))$ and $\Gamma(t)$ and $\eta(t)$ depends on S_{ij}, $\hat{\Sigma}_{ij}(t)$, and $\hat{\Theta}_i(t)$.

STRUCTURE OF OPTIMAL ESTIMATORS

$$\hat{z}_i(t) = L_i\,\hat{x}^{\text{com}}(t) + F_i(t)\,\tilde{I}_i^{\text{loc}}(t)$$

1st term: Common info based estimate

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WHAT ELSE IS NEEDED?

- Iteratively compute $\hat{x}^{\text{com}}(t)$ and $\tilde{I}_i^{\text{loc}}(t)$.
- Iteratively update $\hat{\Sigma}_{ij}(t)$ and $\hat{\Theta}_i(t)$.

Follow Witsenhausen's idea!

STRUCTURE OF OPTIMAL ESTIMATORS

$$\hat{z}_i(t) = L_i \, \hat{x}^{\text{com}}(t) + F_i(t) \, \tilde{I}_i^{\text{loc}}(t)$$

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Iterative update of estimates and covariances

SYSTEM IN TERMS OF DELAYED STATE

Define $w^{(k)}(\ell, t) = \sum_{\tau=t-k}^{t-\ell-1} A^{t-\ell-\tau-1} w(\tau)$

Iterative update of estimates and covariances

SYSTEM IN TERMS OF DELAYED STATE

Define $w^{(k)}(\ell, t) = \sum_{\tau=t-k}^{t-\ell-1} A^{t-\ell-\tau-1} w(\tau)$

Then, $x(t) = A^k x(t-k) + w^{(k)}(0, t) + v_i(t)$

$y_i(t) = C_i A^k x(t-k) + C_i w^{(k)}(0, t) + v_i(t).$

Iterative update of estimates and covariances

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$$y_i(t) = C_i A^k x(t-k) + C_i w^{(k)}(0,t) + v_i(t).$$

Let d^* be the diameter of the graph. Then,

$$I^{\text{com}}(t) = y(1 : t - d^*)$$

$$I_i^{\text{loc}}(t) \subseteq y(t - d^* + 1 : t).$$

We can find a matrix C_i^{loc} and vectors $w_i^{\text{loc}}(t)$ and $v_i^{\text{loc}}(t)$ such that

$$I_i^{\text{loc}}(t) = C_i^{\text{loc}} x(t - d^* + 1) + w_i^{\text{loc}}(t) + v_i^{\text{loc}}(t)$$

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COMPUTING ESTIMATES AND INNOVATION

Define $\hat{x}(t - d^* + 1) = \mathbb{E}[x(t - d^* + 1) \mid I^{\text{com}}(t)]$.

Then, $\hat{x}^{\text{com}}(t) = A^{d^*-1} \hat{x}(t - d^* + 1)$

$$\tilde{I}_i^{\text{loc}}(t) = I_i^{\text{loc}}(t) - C_i^{\text{loc}} \hat{x}(t - d^* + 1)$$

Iterative update of estimates and covariances

SYSTEM IN TERMS OF DELAYED STATE

Define $w^{(k)}(\ell, t) = \sum_{\tau=t-k}^{t-\ell-1} A^{t-\ell-\tau-1} w(\tau)$

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COMPUTING ESTIMATES AND INNOVATION

Define $\hat{x}(t - d^* + 1) = \mathbb{E}[x(t - d^* + 1) \mid I^{com}(t)]$.

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$$\tilde{I}_i^{loc}(t) = I_i^{loc}(t) - C_i^{loc} \hat{x}(t - d^* + 1)$$

KEEPING TRACK OF COVARIANCES

$$\hat{\Sigma}_{ij}(t) = C_i^{loc} P(t - d^* + 1) {C_j^{loc}}^\top + P_{ij}^{w}(t) + P_{ij}^{v}(t).$$

$$\hat{\Theta}_i(t) = A^{d^*-1} P(t - d^* + 1) {C_j^{loc}}^\top + P_i^{\sigma}(t).$$

Extension to infinite horizon setup

ASSUMPTIONS $(A, \sqrt{Q})$ is stabilizable and (A, C) is detectable

OBJECTIVE $\min \limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} c(x(t), \hat{z}(t))$

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STRUCTURE OF OPTIMAL ESTIMATORS

$$\hat{z}_i(t) = L_i\, \hat{x}^{com}(t) + F_i\, \tilde{I}_i^{loc}(t)$$

Note: F_i, Σ_{ij} and Θ_i are time-homogeneous

COMPUTING OPTIMAL GAINS

System of matrix equations: for all $i \in N$,

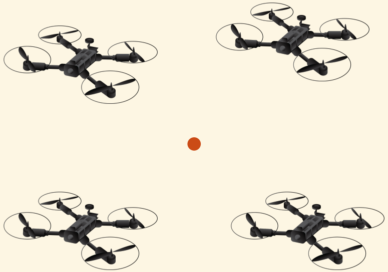
$$\sum_{j\in N} \left[S_{ij} F_j \hat{\Sigma}_{ji} - S_{ij} L_j \hat{\Theta}_j \right] = 0.$$

Example

$x(t) \in \mathbb{R}^4$, $n = 4$ and agent i observes $x_i(t)$.

Per-step cost: $\displaystyle\sum_{i \in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \lambda \|\bar{x}(t) - \bar{z}(t)\|^2$

2-step delay sharing information structure



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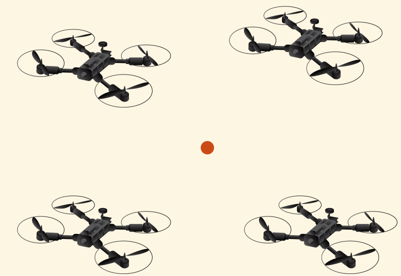

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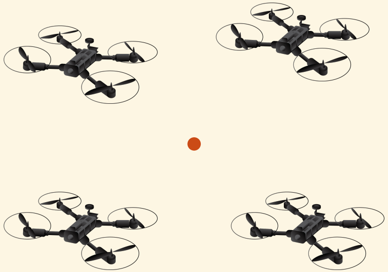

BASELINE STRATEGY $\hat{z}_i(t) = L_i \, \mathbb{E}[x(t) \mid I_i(t)]$

OPTIMAL STRATEGY $\hat{z}_i(t) = L_i x^{\text{com}}(t) + F_i(t) \, \tilde{I}_i^{\text{loc}}(t)$

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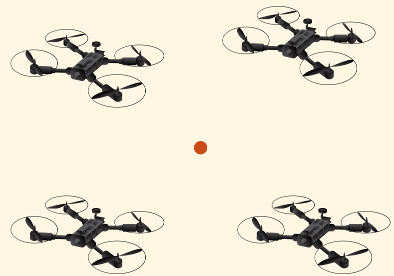
2-step delay sharing information structure



BASELINE STRATEGY	$\hat{z}_i(t) = L_i \, \mathbb{E}[x(t) \mid I_i(t)]$	17.67
OPTIMAL STRATEGY	$\hat{z}_i(t) = L_i x^{\text{com}}(t) + F_i(t) \, \tilde{I}_i^{\text{loc}}(t)$	14.54

17% better

Summary

Summary

System model

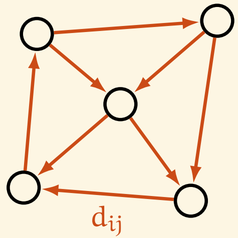
DYNAMICS $x(t+1) = Ax(t) + w(t), \quad w(t) \sim \mathcal{N}(0, Q).$

OBSERVATIONS The system consists of n agents.

$y_i(t) = C_i x(t) + v_i(t), \quad v_i(t) \sim \mathcal{N}(0, R_i).$

INFO STRUCTURE
- Agents communicate over a strongly connected weighted directed graph.
- Edge weight d_{ij} corresponds to link delay.

$$I_i(t) = \{y_i(1:t)\} \cup \Big(\bigcup_{j \in N_i^-} I_j(t - d_{ji}) \Big)$$



Summary

System model

System model (continued)

ESTIMATES — Each agent generates an estimate

$$\hat{z}_i(t) = g_{i,t}(I_i(t))$$

PER-STEP ERROR — Let $\hat{z}(t) = \text{vec}(\hat{z}_1(t), \dots, \hat{z}_n(t))$. Then,

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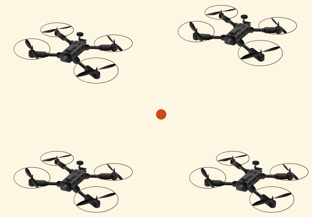
where

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$

OBJECTIVE — Choose a team estimation problem g to

$$\min \mathbb{E}^g \Big[\sum_{t=1}^{T} c(x(t), \hat{z}(t)) \Big]$$



$$\sum_{i \in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \lambda \|\bar{x}(t) - \bar{z}(t)\|^2$$

$$\sum_{i \in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \sum_{i=1}^{n-1} \lambda \|d_i(t) - \hat{d}_i(t)\|^2$$

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Common Information	$I^{\text{com}}(t) = \bigcap_{i \in N} I_i(t),$
Local Information	$I_i^{\text{loc}}(t) = I_i(t) \setminus I^{\text{com}}(t),$
State estimate	$\hat{x}^{\text{com}}(t) = \mathbb{E}[x(t) \mid I^{\text{com}}(t)].$
Local Innovation	$\tilde{I}_i^{\text{loc}}(t) = I_i(t) - \mathbb{E}[I_i^{\text{loc}}(t) \mid I^{\text{com}}(t)].$

Let $\hat{\Sigma}_{ij}(t) = \text{cov}(\tilde{I}_i(t), \tilde{I}_j(t))$.

and $\hat{\Theta}_i(t) = \text{cov}(x(t), \tilde{I}_i(t))$

STRUCTURE OF OPTIMAL ESTIMATORS

$$\hat{z}_i(t) = L_i\, \hat{x}^{\text{com}}(t) + F_i(t)\, \tilde{I}_i^{\text{loc}}(t)$$

1st term: Common info based estimate

2nd term: Local innovation based correction (depends on weight matrix)

Summary

System model

System model (continued)

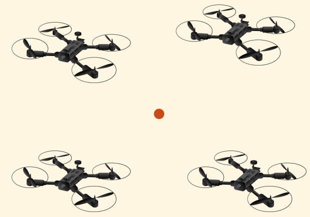
Alternative idea: Common information approach

Example

$x(t) \in \mathbb{R}^4$, $n = 4$ and agent i observes $x_i(t)$.

Per-step cost: $\sum_{i \in N} \|x_i(t) - \hat{z}_i(t)\|^2 + \lambda \|\bar{x}(t) - \bar{z}(t)\|^2$

2-step delay sharing information structure



BASELINE STRATEGY	$\hat{z}_i(t) = L_i\, \mathbb{E}[x(t) \mid I_i(t)]$	17.67
OPTIMAL STRATEGY	$\hat{z}_i(t) = L_i x^{\text{com}}(t) + F_i(t)\, \tilde{I}_i^{\text{loc}}(t)$	14.54

17% better

Summary

System model

System model (continued)

Alternative idea: Common information approach

Example

Motivation for current work

Separation centralized stochastic control, the optimal control action depends on the solution of an estimation problem:

$$\mathbb{E}\Big[\sum_{t=1}^{T}(L(t)\tilde{x}(t)+\hat{z}(t))^{\top}S(t)(L(t)\tilde{x}(t)+\hat{z}(t))\Big]$$

Does the same happen in decentralized control?

In decentralized estimation, is $L\,\mathbb{E}[x(t) \mid I(t)]$ the best estimate?

There is a long history of duality between estimation and control.

Decentralized control is interesting. Ergo, decentralized estimation is interesting.

Decentralized estimation is interesting in it's own right in certain applications.

Decentralized estimation–(Afshari and Mahajan)