An explicit solution of a two user dynamic team

Aditya Mahajan

Dept of ECE
McGill University

September 30, 2010
Allerton

Is dynamic team theory useful?

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Hlyuchj and Gallager, 1981

Although the notion of a dynamic team problem has been around for over 25 years, the class of problems is of sufficient complexity that little progress has been made toward a general solution technique or even in finding general properties of optimal solutions. Hence its value to the multi-access problem does not go much beyond a conceptual level.

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What is the state of the art after 30 years?

Have we made any progress toward a general solution technique to be of any value to the problem that Hlyuchj and Gallager were interested in?

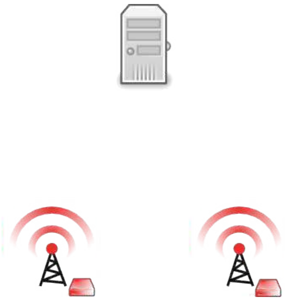
Problem Setup: Two-user multiple access broadcast

Two-users with single slot buffer

- $x_{i,t} \in \{0,1\}$: # packets in buffer
- $a_{i,t} \in \{0,1\}$: # new packet arrivals

$$a_{i,t} \sim \text{Ber}(p_i)$$

- $u_{i,t} \in \{0,1\}$: # transmitted packets



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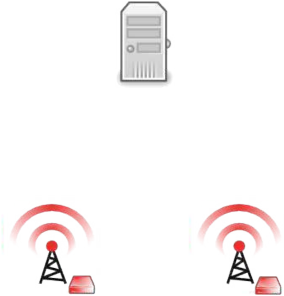

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Multiple access channel

Indicator of successful decoding: $z_t = u_{1,t} \oplus u_{2,t}$

$$x_{i,t+1} = (x_{i,t} - u_{i,t} z_t) \vee a_{i,t}$$

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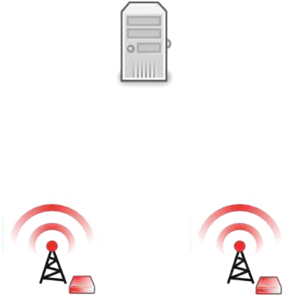

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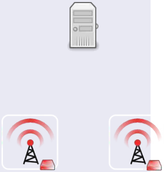
Broadcast channel

z_t is available to the users after unit delay

Problem Setup: Two user multiple access broadcast

Problem (P1)



- *Given*: *arrival rates* p_1 *and* p_2
- *Choose*: *Transmission policies* $(\mathbf{g}_1, \mathbf{g}_2)$ *where* $\mathbf{g}_i = (g_{i,1}, g_{i,2}, \ldots, g_{i,T})$ *and*

$$u_{i,t} = g_{i,t}(x_{i,1:t}, u_{i,1:t-1}, z_{1:t-1})$$

  - *Objective*: *Maximize*

$$\mathbb{E}^{\mathbf{g}_1,\mathbf{g}_2}\left\{\sum_{t=1}^{T} u_{1,t} \oplus u_{2,t}\right\} \quad or \quad \lim_{T\to\infty}\frac{1}{T}\mathbb{E}^{\mathbf{g}_1,\mathbf{g}_2}\left\{\sum_{t=1}^{T} u_{1,t} \oplus u_{2,t}\right\}$$

Literature Overview

Simplest canonical problem in multi-access networks.

- Slotted ALOHA and variants: Provide approximately optimal performance when the number of users is large. Huge literature . . .
- Collision incurs a cost but does not affect the dynamics Schoute, 76, Walrand Varaiya, 79,
- We are interested in the two-user problem in which collision affects the dynamics

Literature overview for Problem (P1)

Symmetric arrival rates

Asymmetric arrival rates

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  - Genie reveals buffer state after a delay
  - Numerical soln
  - upper bound

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- Lot of AI literature . . .
- Hansen et. al. 04
- Bernstein et. al. 05
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- Hansen et. al. 04
  - Numerical algorithm to find optimal soln
  - Out of memory for $T=5$
- Bernstein et. al. 05
  - Heuristic algorithm
  - Controller for size=8
- Szer Charpillet 06
  - Approx. algorithm
  - Out of memory for $T=5$

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Approx algorithms ...
but can only solve the system until $T = 4$

Questions?

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- Optimal soln is known
- The proof is numerical
- Can we provide an analytic proof?

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Asymmetric arrival rates

- Approx algorithms only work for small horizon
- Can we find algorithms that can solve large or infinite horizon problem?

Contributions of this paper

- Provide a dynamic programming decomposition
- The DP has countable state space and finite action space. Easy to use existing algorithms to find numerical solution for large or infinite horizon setups
- For symmetric arrival rates, find an analytic soln to the DP.

Problem Setup: Two user multiple access broadcast

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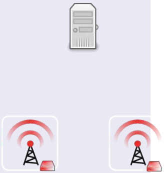

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Solution Outline

Transmission policy

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Feedback ≡ control sharing

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$u_{i,t} = g_{i,t}(x_{i,t}, \pi_{1,t}, \pi_{2:t})$
where
$\pi_{i,t} = \Pr\left(x_{i,t} = 1 | u_{1,1:t-1}, u_{2,1:t-1}\right)$

Solution Outline (cont)

Dynamic Program

$$V_{T+1}(\pi_1, \pi_2) = 0$$

and for $t = T, T-1, \ldots, 1$

$$V_t(\pi_1, \pi_2) = \max\{W_{10,t}(\pi_1, \pi_2), W_{01,t}(\pi_1, \pi_2), W_{11,t}(\pi_1, \pi_2)\}$$

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Reachability Analysis

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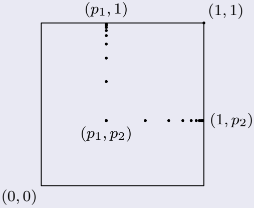

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Sufficient statistic for common information

$$u_{i,t} = g_{i,t}(x_{i,t}, u_{1,1:t-1}, u_{2,1:t-1})$$

- Common information: $(u_{1,1:t-1}, u_{2,1:t-1})$
- Private information: $x_{i,t}$

A special case of Mahajan, Nayyar, Teneketzis, 2008

Same solution approach (using the notion of a coordinator) applies

Sufficient statistic for common information (cont)

Coordinator of the two users

- Observation of coordinator: common information

$$(u_{1,1:t-1}, u_{2,1:t-1})$$

- Action of the coordinator: partial functions $(\gamma_{1,t}, \gamma_{2,t})$ s.t.

$$u_{i,t} = \gamma_{i,t}(x_{i,t})$$

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- For ease of notation, let $\varphi_{i,t} = \gamma_{i,t}(1)$. Then

$$u_{i,t} = \varphi_{i,t} x_{i,t}$$

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- Think of $(\varphi_{1,t}, \varphi_{2,t})$ as the control action of the coordinator.

Optimization problem at the coordinator

Problem (P2)

- *Given: arrival rates p_1 and p_2*
- *Choose: Coordination policy $\mathbf{h} = (h_1, h_2, \dots, h_T)$ where*

$$(\varphi_{1,t}, \varphi_{2,t}) = h_t(u_{1,1:t-1}, u_{2,1:t-1}, \varphi_{1,1:t-1}, \varphi_{2,1:t-1})$$

- *Objective: Maximize*

$$\mathbb{E}^{\mathbf{h}}\left\{ \sum_{t=1}^{T} u_{1,t} \oplus u_{2,t} \right\} \quad \text{or} \quad \lim_{T\to\infty} \frac{1}{T} \mathbb{E}^{\mathbf{h}}\left\{ \sum_{t=1}^{T} u_{1,t} \oplus u_{2,t} \right\}$$

Sufficient statistic for common information (cont)

Proposition

Problem (P1) and (P2) are equivalent.

Proof.

- Any transmission policy $(\mathbf{g}_1, \mathbf{g}_2)$ for (P1) can be implemented in (P2) by choosing

$$\varphi_{i,t} = g_{i,t}(1, u_{1,1:t-1}, u_{2,1:t-1})$$

resulting in identical realization of all system variables.

- Any coordination policy $\mathbf{h}$ for (P2) can be implemented in (P1) by choosing

$$g_{i,t}(x_{i,t}, u_{1,1:t-1}, u_{2,1:t-1}) = \varphi_{i,t} x_{i,t}$$

where $\varphi_{i,t}$ is recursively chosen according to

$$(\varphi_{1,t}, \varphi_{2,t}) = h_t(u_{1,1:t-1}, u_{2,1:t-1}, \varphi_{1,1:t-1}, \varphi_{2,1:t-1})$$

□

Sufficient statistic for common information (cont)

Definition

$$\pi_{i,t} = \Pr\left(x_{i,t} = 1 \,\middle|\, \begin{array}{c} u_{1,1:t-1}, u_{2,1:t-1} \\ \varphi_{1,1:t-1}, \varphi_{2,1:t-1} \end{array}\right)$$

Proposition

In (P2), restricting attention to coordination policies of the form

$$(\varphi_{1,t}, \varphi_{2,t}) = h_t(\pi_{1,t}, \pi_{2,t})$$

is without loss. Therefore, in (P1) restricting attention to transmission policies of the form

$$u_{i,t} = g_{i,t}(x_{i,t}, \pi_{1,t}, \pi_{2,t}$$

is without loss.

Sufficient statistic for common information (cont)

Proof.

- $(\pi_{1,t}, \pi_{2,t})$ is a controlled Markov process with control action $(\varphi_{1,t}, \varphi_{2,t})$.
- Expected conditional reward

$$
\begin{aligned}
&\mathbb{E}[u_{1,t} \oplus u_{2,t} | u_{1,1:t-1}, u_{2,1:t-}, \varphi_{1,1:t}, \varphi_{2,1:t}] \\
&\quad = \pi_{1,t}\varphi_{1,t}(1 - \pi_{2,t}\varphi_{2,t}) + (1 - \pi_{1,t}\varphi_{1,t})\pi_{2,t}\varphi_{2,t} \\
&\quad = \mathbb{E}[u_{1,t} \oplus u_{2,t} | \pi_{1,t}, \pi_{2,t}, \varphi_{1,t}, \varphi_{2,t}]
\end{aligned}
$$

□

Solution Outline (cont)

Dynamic Program

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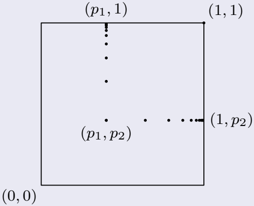
Reachability Analysis

The reachable set of (π_1, π_2) is countable.



Dynamic Program

- DP follows immediately from the fact that $(\pi_{1,t}, \pi_{2,t})$ is a controlled Markov process.
- By the same argument, the DP naturally extends to infinite horizon setup.

Reachability Analysis

- Let A_i be an operator from $[0,1]$ to $[0,1]$ such that for any $\pi \in [0,1]$
$$A_i\pi = 1 - (1 - p_i)(1 - \pi)$$

Evolution of info state

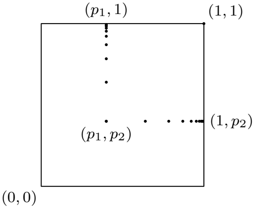
1. When $(\varphi_{1,t}, \varphi_{2,t}) = (0,0)$, $(\pi_{1,t+1}, \pi_{2,t+1}) = (A_1\pi_{1,t}, A_2\pi_{2,t})$.
2. When $(\varphi_{1,t}, \varphi_{2,t}) = (1,0)$, $(\pi_{1,t+1}, \pi_{2,t+1}) = (p_1, A_2\pi_{2,t})$.
3. When $(\varphi_{1,t}, \varphi_{2,t}) = (0,1)$, $(\pi_{1,t+1}, \pi_{2,t+1}) = (A_1\pi_{1,t}, p_2)$.
4. When $(\varphi_{1,t}, \varphi_{2,t}) = (1,1)$,
$$(\pi_{1,t+1}, \pi_{2,t+1}) = \begin{cases} (1,1) & \text{if } x_{1,t} = x_{2,t} = 1 \\ (p_1, p_2) & \text{otherwise} \end{cases}$$

Reachability Analysis (cont)

Reachable Set

Suppose the system starts in state $(\pi_1, \pi_2) = (p_1, p_2)$. Then the reachable set of (π_1, π_2) is

$$S = \{(1,1), (p_1, 1), (1, p_2), (p_1, p_2)\} \bigcup \{(A_1^n p_1, p_2), (p_1, A_2^n p_2), : n \in \mathbb{N}\}$$



Reachability Analysis (cont)

- The reachable set of $(\pi_{1,t}, \pi_{2,t})$ is countable.
- Thus, the inifnite horizon DP has countable state space and finite action space
- Stanard techniques to numerically solve such DP (e.g. Sennot, 97 , Leizarowitz Schwartz, 07)
- Contrast this with earlier attempt to obtain a numerical solution for this problem.

Symmetric arrivals

- Optimal coordination policy is symmetric $h(\pi_1, \pi_2) = h(\pi_2, \pi_1)$

Some definitions

- Let $\tau \approx 0.38196$ be the root of $x = (1 - x)^2$ that lies in $[0, 1]$.

Symmetric arrivals

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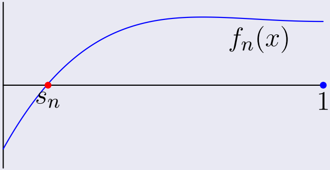
Some definitions

- Let $\tau \approx 0.38196$ be the root of $x = (1-x)^2$ that lies in $[0,1]$.
- Let $f_n(x) = 1 + (1-x)^2 - (3+x)(1-x)^{n+1}$



  and s_n denote the root of $f_n(x)$ that is between $[0,1]$.
- $s_0 > \tau > s_1 > s_2 > \cdots 0$

Symmetric arrivals

Theorem

An optimal policy of the infinite horizon variant of (P2) is:

- round-robin policy for $p \geq \tau$

$$h^*(\pi_1, \pi_2) = \begin{cases} (1,0) & \text{if } \pi_1 > \pi_2, \\ (0,1) & \text{if } \pi_1 < \pi_2, \\ (1,0) \text{ or } (0,1) & \text{if } \pi_1 = \pi_2. \end{cases}$$

- transmit if you have a packet policy for $p < \tau$

$$h^*(\pi_1, \pi_2) = \begin{cases} (1,1) & \text{if } \pi_1 \leq A^m p, \pi_2 \leq A^m p, \\ (1,0) & \text{if } \pi_1 > \pi_2, \pi_1 > A^m p \\ (0,1) & \text{if } \pi_1 < \pi_2, \pi_2 > A^m p \\ (1,0) \text{ or } (0,1) & \text{if } \pi_1 = \pi_2 = 1. \end{cases}$$

where m is s.t. $s_{m+1} \leq p \leq s_m$.

Symmetric arrivals

Theorem

The average reward per unit time for the infinite horizon variant of (P2) is

$$J^* = \begin{cases} p[1-(2p^2-1)/D(p)] & \text{if } p \leq s_1, \\ (1-\bar{p}^2) & \text{if } s_1 \leq p; \end{cases}$$

where $\bar{p} = 1 - p$ and $D(p) = 1 + p^2 + p^3$.

Symmetric arrivals

Proof

Guess the form of the value function and verify!

1. When $p \geq s_1$,

$$\begin{aligned} v(p, A^n p) = v(A^n p, p) &= (1 - \bar{p}^{n+1}), \quad n > 1 \\ v(p, 1) = v(1, p) &= 1, \\ v(1,1) &= (1 + \bar{p}^2), \\ v(p,p) &= p \end{aligned}$$

Symmetric arrivals

Proof (cont)

Guess the form of the value function and verify!

2. When $s_{m+1} \leq p < s_m$, $m \in \mathbb{N}$

$$v(p,1) = v(1,p) = p[1 - f_0(p)/D(p)],$$
$$v(1,1) = 1,$$
$$v(p,p) = f_1(p)/D(p),$$
$$v(A^n p, p) = v(p, A^n p) = \begin{cases} c_*(n) & \text{if } n \leq m, \\ c^*(n) & \text{if } n > m \end{cases}$$

where

$$c_*(n) = \frac{\bar{p}}{p}(1 - \bar{p}^n)J^* + \bar{p}^{n+1} - \bar{p} + v(p,p),$$
$$c^*(n) = (1 - \bar{p}^{n+1}) + c_*(1) - v(1,p)$$

Symmetric arrivals

Proof

Guess the form of the value function and verify!

- Rest is just a matter of elementary (but tedious) algebra.
- The important point is that once we have a dynamic program, optimality of a particular policy can be checked systematically.

Proof

Guess the form of the value function and verify!

- Rest is just a matter of elementary (but tedious) algebra.
- The important point is that once we have a dynamic program, optimality of a particular policy can be checked systematically.
- We also need to guess the differential reward functions for the non-optimal actions. In general, this can be difficult. But, we exploit the symmetry and the fact that state space is countable.

Conclusion

Contributions

- An interesting example of two-user dynamic team that can be solved explicitly.
- For symmetric arrivals, identified the optimal policy analytically. The previous proof of optimality involved numerically solving a genie aided upper bound.
- For asymmetric arrivals, identified a DP with countable state space and finite action space. Earlier attempts for a numerical solution could only solve finite horizon problems with $T = 4$.

Conclusion

Future work

- We are missing a structural result:
  Each user gets a transmission opportunity $\varphi_{i,t} = 1$, at least once in two consecutive time slots
- The optimal policy satisfies this property.
- If we can prove this upfront, the DP will be much simpler (finite state and finite action spaces).

Thank You