Mean-field approximation for large-population beauty-contest games

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Presentation for IEEE Conference on Decision and Control (CDC) 2021

December 16, 2021

Presentation overview

1. Beauty contest games.
2. Specification of the game.
3. Characterization of Bayesian Nash equilibrium.
4. Mean-field approximation.
5. Simulation Results.
6. Conclusions.

Beauty contest games

Classical beauty contest games

- Introduced by Keynes in 1936.
- Describes beauty contest where judges are rewarded for selecting most popular faces.

Features

- Strategic games.
- Players make a choice that is close to a certain aggregate choice of the group.

Applications

- Trading decisions in financial markets.
- Social value of information.

Model

We consider general sum Bayesian game with $n \in N$ players trying to estimate θ where $\theta \sim \mathcal{N}(0,1)$ from observations. The players have access to both private and common observations.

Common observation

$y_0 = \alpha_0\theta + v_0$, where $\alpha_0 \in [0,1]$ and $v_0 \sim \mathcal{N}(0, \sigma_0^2)$

Private observation

$y_i = \alpha_i\theta + v_i$, where $\alpha_i \in [0,1]$ and $v_i \sim \mathcal{N}(0, \sigma^2) \forall i \in N$

Cost incurred by player i

$$c_i = (1-\lambda_i)(\theta - u_i)^2 + \lambda_i(u_i - \rho_i\bar{u})^2$$

where, $\lambda_i \in [0,1]$, $\rho_i \in \mathbb{R}$ and $\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i$.

Model

Interpretation of λ_i

λ_i weights the two quadratic terms and the relative trade-off of the accuracy of players estimate with its "popularity".

Interpretation of ρ_i

$\rho_i \in \mathbb{R}$ might represent:

- Degree of "bullishness" of an asset in financial context.
- Degree of "polarization" when evaluating a political issue.

Player parameters

The parameters for player i is represented by $\phi_i = (\alpha_i, \rho_i, \lambda_i)$. The parameter of all players is denoted by $\phi = (\alpha_0, \phi_1, \ldots, \phi_n)$.

Bayesian Nash equilibrium in affine strategies

Theorem 1

There exits a Bayesian Nash Equilibrium (BNE) of the form

$$g(y_0, y_i, \phi) = a_i y_0 + b_i y_i \quad \forall i \in N$$

where a_i and b_i are obtained by solving the following system of linear equations.

$$A\mathrm{a} + \bar{B}\mathrm{b} = \eta, \quad B\mathrm{b} = \kappa$$

The BNE is unique if both A and B are invertible.

Bayesian Nash equilibrium in affine strategies

System of equations: $A\text{a} + \bar{B}\text{b} = \eta, \quad B\text{b} = \kappa$

$$A_{i,j} = \begin{cases} \Lambda_i & \text{if i=j} \\ -\bar{\Lambda}_i \text{ if } i \neq j \end{cases} \quad B_{i,j} = \begin{cases} \Lambda_i \text{ if } i = j \\ -\bar{\Lambda}_i K_i \alpha_j \text{ if } i \neq j \end{cases}$$

$$\bar{B}_{i,j} = \begin{cases} 0 \text{ if } i = j \\ -\bar{\Lambda}_i H_i \alpha_j \text{ if } i \neq j \end{cases}$$

$$\eta = \text{vec}((1-\lambda_1)H_1, \ldots, (1-\lambda_n)H_n)$$

$$\kappa = \text{vec}((1-\lambda_1)K_1, \ldots, (1-\lambda_n)K_n)$$

$$\Lambda_i = (1-\lambda_i) + \lambda_i(1-\frac{\rho_i}{n})^2, \quad \bar{\Lambda} = \lambda_i \frac{\rho_i}{n}(1-\frac{\rho_i}{n})$$

$$H_i = \frac{\alpha_0 \sigma^2}{\alpha_0^2 \sigma^2 + \alpha_i^2 \sigma_0^2 + \sigma_0^2 \sigma^2}, \quad K_i = \frac{\alpha_i \sigma^2}{\alpha_0^2 \sigma^2 + \alpha_i^2 \sigma_0^2 + \sigma_0^2 \sigma^2}$$

BNE with Homogeneous Players

Symmetric BNE

The BNE obtained is symmetric and of the form

$$ay_i + b,$$

where a and b is given by

$$a = \frac{(1-\lambda)H + (n-1)\bar{\Lambda}H\alpha b}{\Lambda(n-1)\bar{\Lambda}}; \quad b = \frac{(1-\lambda)K}{\Lambda(n-1)\bar{\Lambda}K\alpha}$$

Mean-field approximation

Motivation

- BNE derived with the assumption that parameters are common knowledge which is unlikely to hold for large number of players.
- Solving n system of linear equations for large values of n can get computationally expensive.

Approximation

Compute the mean-field limit of the game assuming $\lim n \to \infty$.

Mean-field approximation

Parameter $\phi = (\alpha_0, \phi_1, \ldots, \phi_n)$ modelled as realizations of random allocations

Assumptions

- α_i, λ_i have support $[0, 1]$.
- ρ_i have a finite support.
- $\alpha_i, \rho_i, \lambda_i$ are independent and identically distributed across players and independent of α_0.

Characterization of mean-field equilibrium

Theorem 2

An ε-BNE of the n player beauty contest game with $\varepsilon \in \mathcal{O}(1/\sqrt{n})$ and $\bar{\lambda}\bar{\rho}\bar{L} \neq 1$, $\bar{\lambda}\bar{\rho} \neq 1$exists and is given by

$$\bar{g}(y_0, y_i, \phi_i) = \left[1 - \lambda_i + \lambda_i \rho_i \bar{M}\right](H_i y_0 + K_i y_i) + \rho_i \lambda_i \bar{a} y_0$$

where,

$$\bar{M} = \frac{(1-\bar{\lambda})\bar{L}}{1-\bar{\lambda}\bar{\rho}\bar{L}}; \quad \bar{a} = \frac{\left[(1-\bar{\lambda}) + \bar{\lambda}\bar{\rho}\bar{M}\right]\bar{H}}{1-\bar{\lambda}\bar{\rho}}$$

and $\bar{\lambda}, \bar{\rho}, \bar{H}$ are the mean of λ_i, ρ_i, H_i and $\bar{L} = \mathbb{E}_{\alpha_i, \alpha_0}[K_i \alpha_i]$.

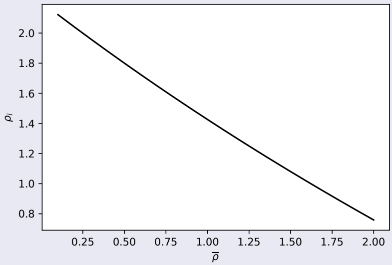
Implications: Players only need to know the parameter distribution, to obtain the mean-field strategy which is independent of n.

Effect on aggregate population behavior on individual behavior

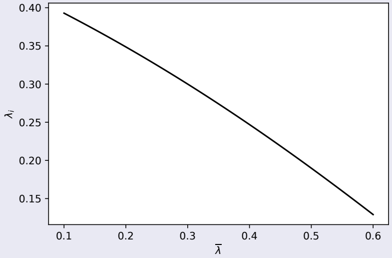
Parameter selection

$\sigma_0 = \sigma = 1,\ \alpha_0 = 0.5,\ \alpha_j \sim \texttt{unif}[0,1], j \in N$
$\bar{H} = 0.288,\ \bar{L} = 0.184,\ \bar{\rho} = 1.25,\ \bar{\lambda} = 0.3$
$\phi_i^\circ = (\alpha_i^\circ, \lambda_i^\circ, \rho_i^\circ) = (0.5, 0.3, 1.25)$

Scenario 1



Scenario 2



Conclusion

- Present a static general sum beauty contest games and identify BNE with in the class of affine strategies.
- Obtain mean-field approximation for a large player system and show that the mean-field strategy is an $\varepsilon-$Nash equilibrium for the n player beauty contest games.
- Future work includes decision making in dynamic settings where the decision of the players evolve over time.

Thank You