Error Exponents of Compound Channel with Feedback

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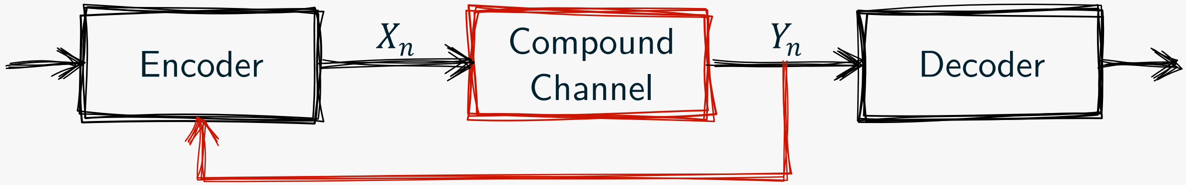
Yale University

ISIT 2010

Problem Setup



Compound channel

- Channel is memoryless

$$\mathbb{P}(Y_n \mid X^n, Y^{n-1}) = Q_\circ(Y_n \mid X_n)$$

- Channel is not known completely

$$Q_\circ \in \boldsymbol{\mathcal{Q}} := \{Q_1, Q_2, \ldots, Q_L\} \qquad \text{(Family of DMCs)}$$

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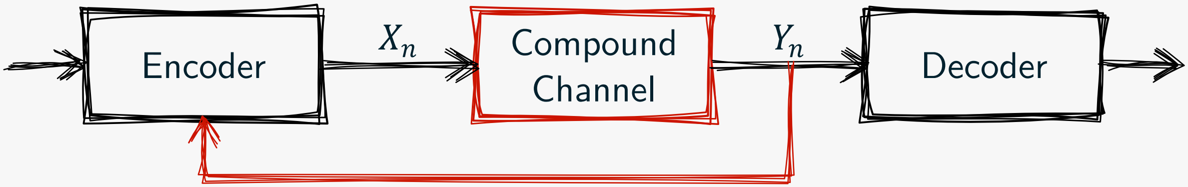

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- Variable length communication
  - Capacity? (easy)
  - Error Exponents? (this talk)

Opportunistic Capacity

- Notation

  For a (variable length) coding scheme S

  - $P_\ell^{(S)}$ = Prob of error when $Q_\circ = Q_\ell$
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Opportunistic Achievability (main idea)

A rate $(R_1, R_2, \ldots, R_L)$ is achievable if $\exists$ a sequence of coding schemes $\{S_n\}_{n\in\mathbb{N}}$ such that for any $\varepsilon > 0$, $\exists n_\circ(\varepsilon)$ so that for all $n > n_\circ(\varepsilon)$

$$P_\ell^{(S_n)} < \varepsilon \quad \text{and} \quad R_\ell^{(S_n)} > R_\ell - \varepsilon \quad \text{for all } \ell = 1, 2, \ldots, L$$

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Opportunistic Capacity: Union of all achievable rates.

$$\mathcal{C}_F(\mathcal{Q}) = \{(R_1, R_2, \ldots, R_L) : 0 \le R_\ell \le C_{Q_\ell},\ \ell = 1, 2, \ldots, L\}$$

where C_{Q_ℓ} = capacity of DMC Q_ℓ.

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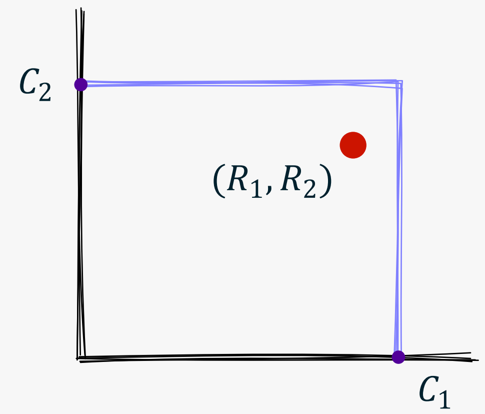

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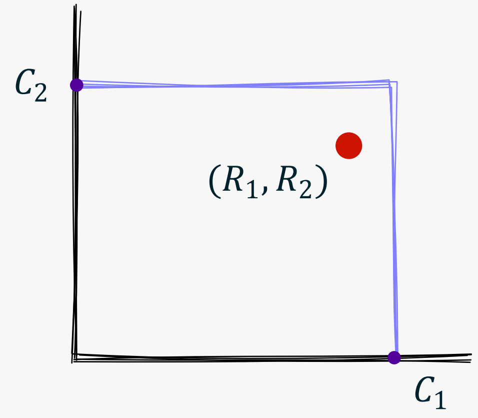

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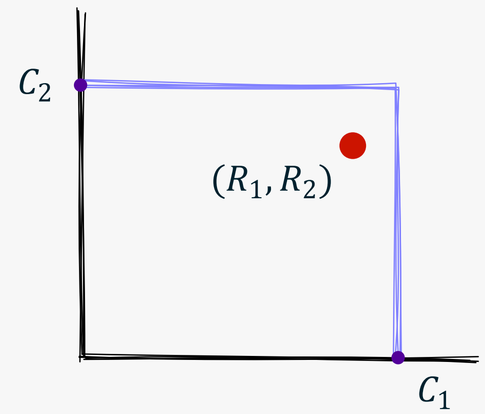

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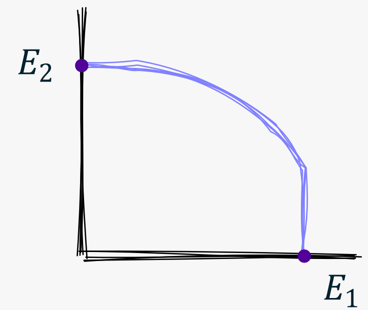

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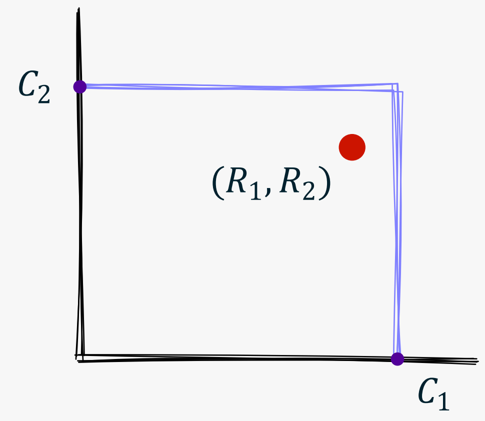

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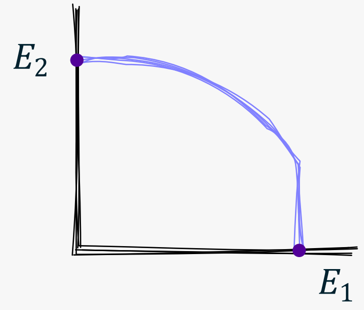

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Error Exponent Region

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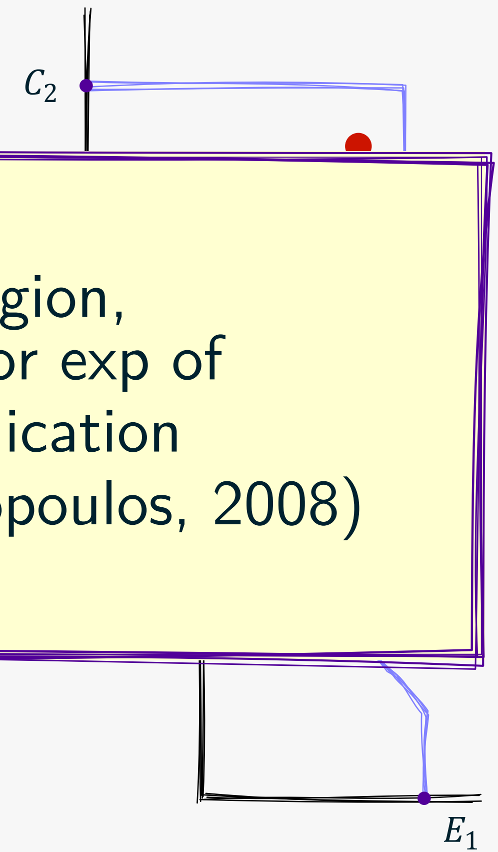
Since capacity is a region,
error exp behave like error exp of
multi-terminal communication
(cf. Weng, Pradhan, Anastasopoulos, 2008)

schemes



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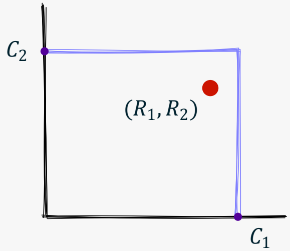
Flavor of the results



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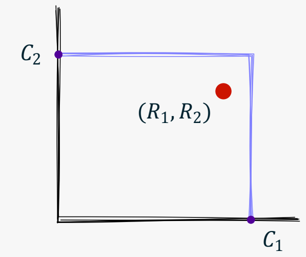

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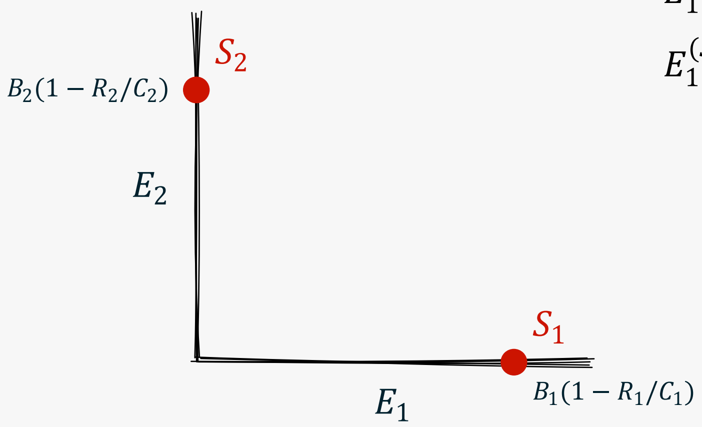

$E_1^{(S_1)} = B_1(1 - R_1/C_1)$ $\quad$ $E_2^{(S_1)} = \text{small}$

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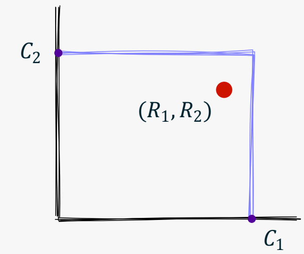

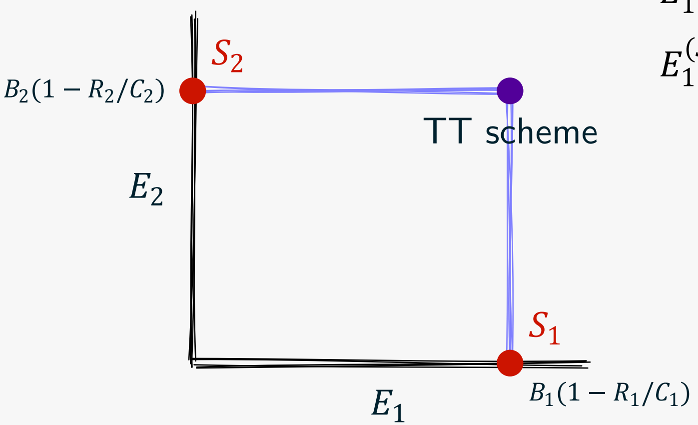

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Tchamkerten and Telatar, 2006

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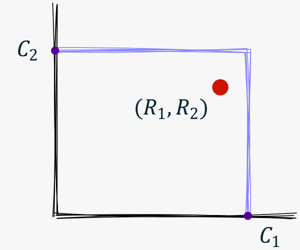

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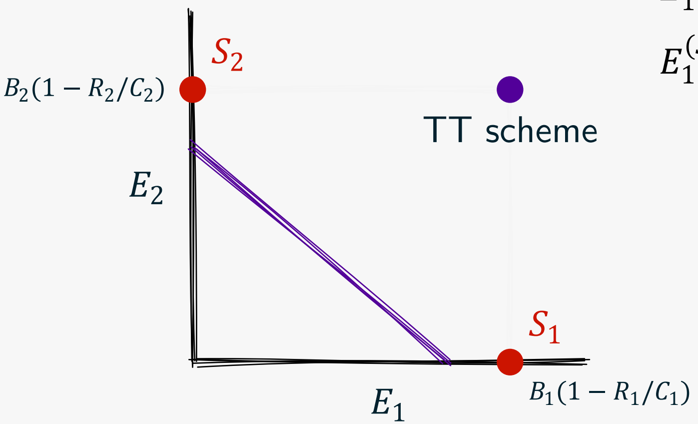

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- Restricted to $R_\ell/C_\ell =$ constant

- Our result: Propose a simple training based scheme such that

$$E_\ell \geq \lambda_{\boldsymbol{Q}} B_\ell \left(1 - \frac{R_\ell}{C_\ell}\right), \qquad \ell = 1, 2$$

Outline

- Literature overview
  - Burnashev exponent, Yamamoto-Itoh scheme
  - Tchamkerten-Telatar result
- Variable length coding scheme

  Transmit a compound message
- An achievable scheme

  A training based variation of Yamamoto-Itoh scheme
- Example

  $\{\mathrm{BSC}_p, \mathrm{BSC}_{1-p}\}$. Non-obvious channel estimation

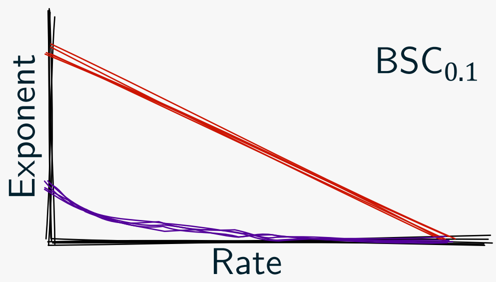
Literature Overview

DMC with Feedback

- Variable length coding significantly boosts the error exponents [Burnashev, 1976]

$$E = B_Q \left(1 - \frac{R}{C}\right)$$

where $B_Q = \max_{x_A, x_R \in \mathcal{X}} D(Q(\cdot|x_A) \| Q(\cdot|x_R))$



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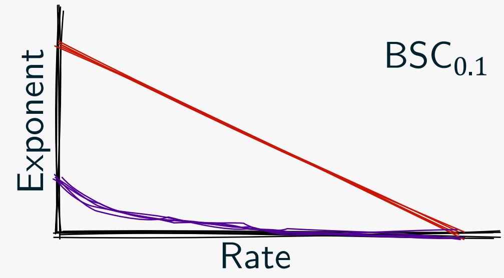

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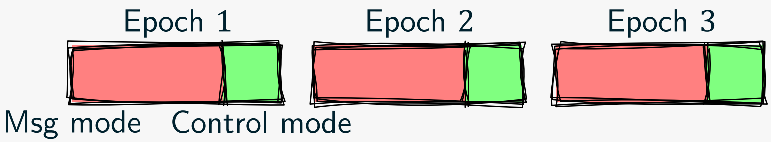

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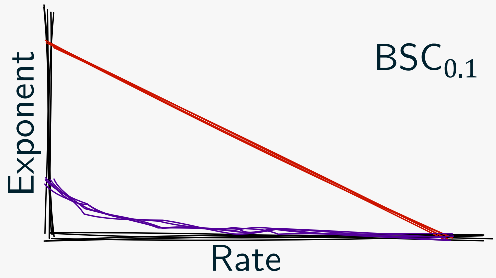

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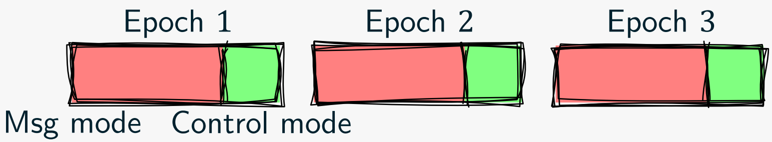

- Any scheme that achieves Burnashev exponent must have a control phase [Berlin *et. al*, 2009]

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[Tchamkerten Telatar, 2006a]

The slope of the error exponent goes to zero at rates near capacity

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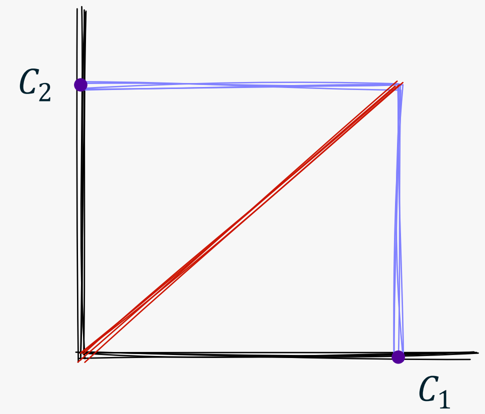

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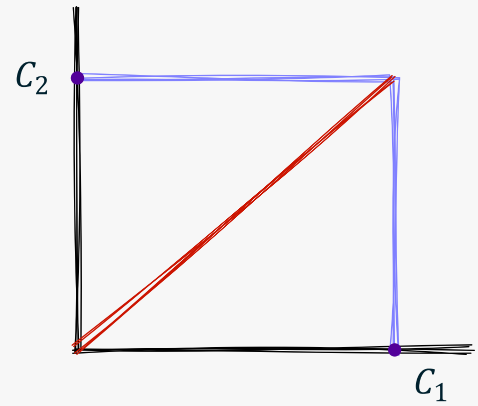

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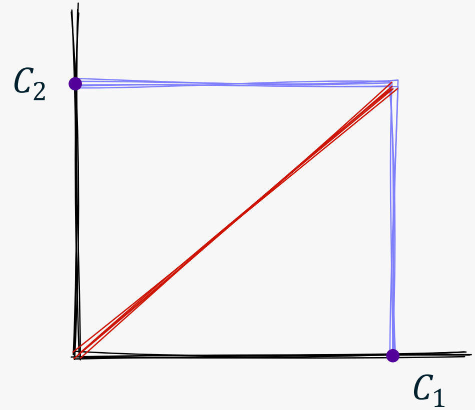

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Variable length coding over compound channel with feedback

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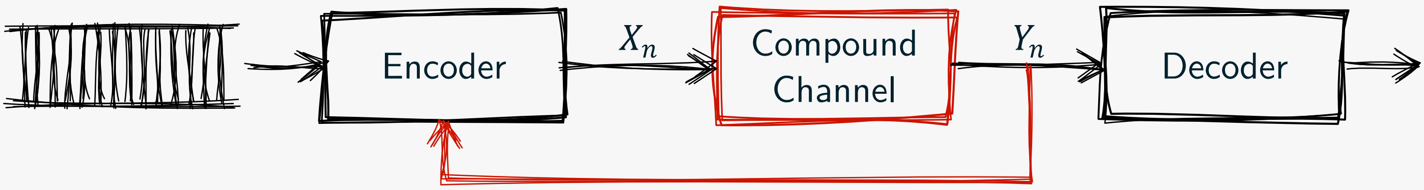

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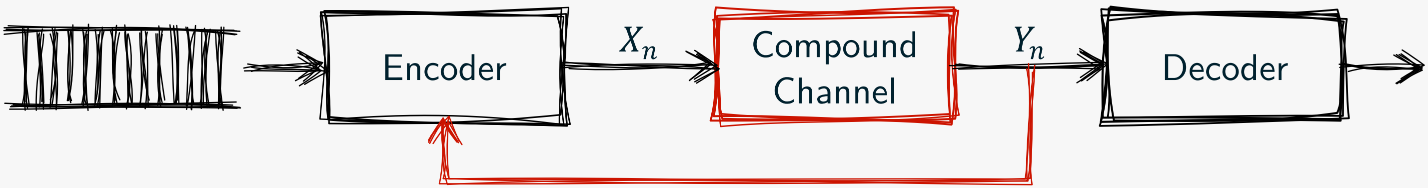

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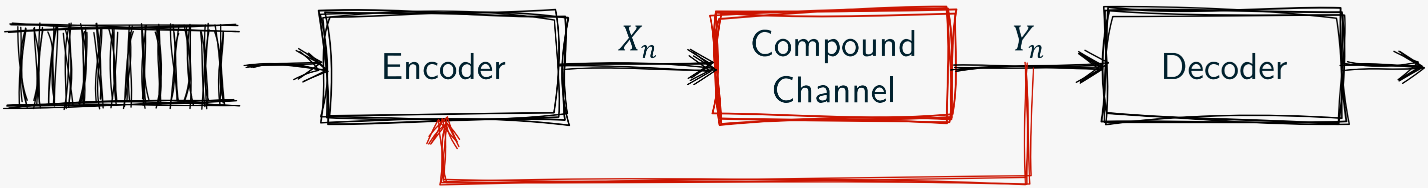

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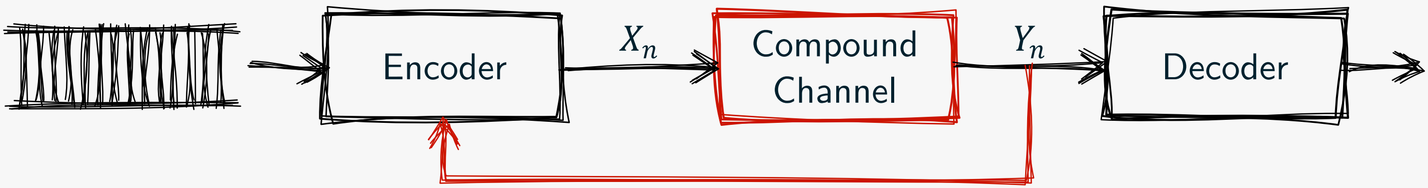

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- Decoding scheme: at stopping time τ, choose

$$(\hat{L}, \hat{W}) = g_\tau(Y_1, Y_2, \ldots, Y_\tau)$$

The coding scheme

Coding scheme is a tuple $(\boldsymbol{M}, \boldsymbol{f}, \boldsymbol{g}, \tau)$

- Compound message size
  $\boldsymbol{M} = (M_1, M_2, \dots, M_L)$

- Compound message alphabet
  $\boldsymbol{\mathcal{M}} = \prod_{\ell=1}^{L} \{1, 2, \dots, M_\ell\}$

- Encoding strategy $\boldsymbol{f} = (f_1, f_2, \dots)$

  $f_t : \boldsymbol{\mathcal{M}} \times \boldsymbol{\mathcal{Y}}^{t-1} \mapsto \boldsymbol{\mathcal{X}}$

- Decoding strategy $\boldsymbol{g} = (g_1, g_2, \dots)$

  $$g_t : \boldsymbol{\mathcal{Y}}^{t-1} \mapsto \bigcup_{\ell=1}^{L} \{(\ell, 1), (\ell, 2), \dots, (\ell, M_\ell)\}$$

- Stopping time τ with respect to filtration $\{2^{\boldsymbol{\mathcal{Y}}^t}, t \in \mathbb{N}\}$

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- Stopping time τ with respect to filtration $\{2^{\mathcal{Y}^t}, t \in \mathbb{N}\}$

- Rate $\boldsymbol{R} = (R_1, R_2, \ldots, R_L)$
  $$R_\ell = \frac{\mathbb{E}_\ell[\log_2 M_{\hat{L}}]}{\mathbb{E}_\ell[\tau]}$$
- Probability of error $\boldsymbol{P} = (P_1, P_2, \ldots, P_L)$
  $$P_\ell = \mathbb{P}_\ell(\hat{W} \neq W_{\hat{L}})$$

Capacity and Error Exponents

- Achievable Rate: A rate $\boldsymbol{R} = (R_1, R_2, \ldots, R_L)$ is achievable if there exists a sequence of coding schemes $(\boldsymbol{M}^{(n)}, \boldsymbol{f}^{(n)}, \boldsymbol{g}^{(n)}, \tau^{(n)})$, $n \in \mathbb{N}$, such that

  1. $\lim_{n\to\infty} \mathbb{E}_\ell[\tau^{(n)}] = \infty$, $\ell = 1, 2, \ldots, L$
  2. For any $\varepsilon > 0$, $\exists n_\circ(\varepsilon)$ s.t. $\forall n \geq n_\circ(\varepsilon)$

$$P_\ell^{(n)} < \varepsilon \quad \text{and} \quad R_\ell^{(n)} \geq R_\ell - \varepsilon, \qquad \ell = 1, 2, \ldots, L$$

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$$P_\ell^{(n)} < \varepsilon \quad \text{and} \quad R_\ell^{(n)} \geq R_\ell - \varepsilon, \qquad \ell = 1, 2, \ldots, L$$

- Capacity: Union of all achievable rates

- Error Exponent: Given a sequence of coding schemes $(\boldsymbol{M}^{(n)}, \boldsymbol{f}^{(n)}, \boldsymbol{g}^{(n)}, \tau^{(n)})$, that achieve a rate $\boldsymbol{R}$, the error exponent is given by

$$E_\ell = \lim_{n\to\infty} -\frac{\log P_\ell^{(n)}}{\mathbb{E}_\ell[\tau^{(n)}]}$$

- Error Exponent Region: Union of all error exponents.

- Reliability: Pareoto frontier of Error Exponent region

Main Result

- Opportunistic Capacity

$$\mathcal{C}_F(\mathcal{Q}) = \{(R_1, \ldots, R_L) : 0 \leq R_\ell < C_\ell, \ell = 1, \ldots, L\}$$

where C_ℓ is the capacity of DMC Q_ℓ.

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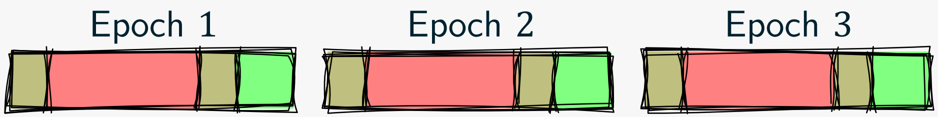
  - At rate $\boldsymbol{R} = (R_1, \ldots, R_L)$, the error exponent is

$$E_\ell \ge \frac{T^c_\ell}{T^c_\ell + B_{Q_\ell}} B_{Q_\ell} \left(1 - \frac{R_\ell}{C_\ell}\right)$$

where $B_{Q_\ell} = \max_{x_A, x_R \in \mathcal{X}} D(Q_\ell(\cdot|x_A) \| Q_\ell(\cdot|x_R))$

Achievable Scheme for $(R_1, \ldots, R_L)$



Parameterized by $n \in \mathbb{N}$. Multiple epochs. Each epoch has four phases

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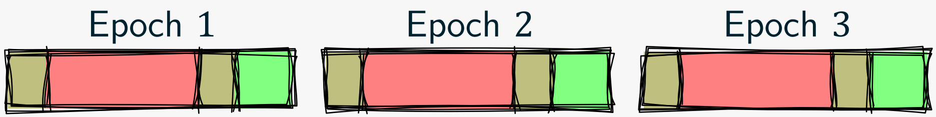

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1. Training phase: length $\beta_1^{(n)} n$.
   Channel estimate $\hat{L}_m^{(k,n)}$.
2. Message phase: length $\beta_2^{(n)}(\hat{L}_m) n$
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4. Control phase: length $\beta_4^{(n)}(\hat{L}_c) n$.

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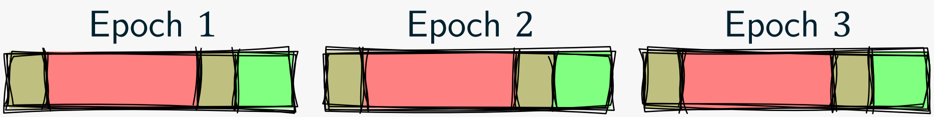

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- Estimation rules: $\hat{\theta}_m^{(n)}$ for phase one, $\hat{\theta}_c^{(n)}$ for phase three.
- Training exponents: $(T_1^m, \ldots, T_L^m)$ and $(T_1^c, \ldots, T_L^c)$ respectively

$$T_\ell = \mathbb{P}_\ell(\text{Channel estimation is wrong})$$

Choice of parameters

Let $\kappa_\ell = \frac{T_\ell^c}{B_{Q_\ell}}$ and $\gamma_\ell = \frac{R_\ell}{C_\ell}$

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$$\beta_3^{(n)} \uparrow \frac{(1-\gamma^\star)}{(1+\kappa^\star)}, \qquad \beta_4^{(n)}(\ell) \uparrow \kappa_\ell \beta_3^{(n)},$$

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$$\beta_1^{(n)} \downarrow 0, \qquad \beta_2^{(n)}(\ell) \downarrow \alpha_\ell\gamma_\ell, \qquad \mathbb{E}_\ell[\beta_1^{(n)} + \beta_2^{(n)}(\ell)] \downarrow \alpha_\ell\gamma_\ell$$

$$\beta_3^{(n)} \uparrow \underbrace{\frac{(1-\gamma^\star)}{(1+\kappa^\star)}}_{=\alpha_\ell\frac{(1-\gamma_\ell)}{(1+\gamma_\ell)}}, \qquad \beta_4^{(n)}(\ell) \uparrow \kappa_\ell\beta_3^{(n)}, \qquad \mathbb{E}_\ell[\beta_3^{(n)} + \beta_3^{(n)}(\ell)] \uparrow \alpha_\ell(1-\gamma_\ell)$$

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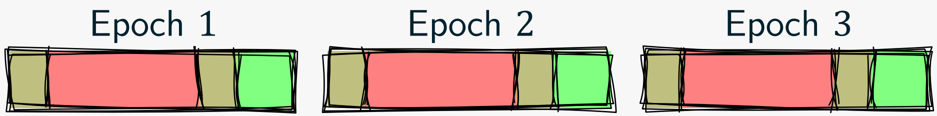
- Expected length of an epoch

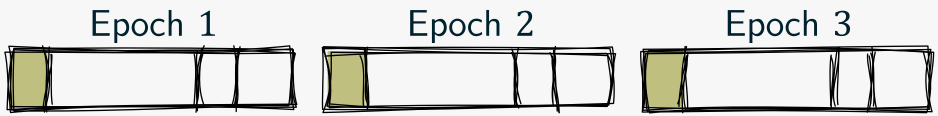
$$\mathbb{E}_\ell[\beta_1^{(n)} + \beta_2^{(n)}(\ell) + \beta_3^{(n)} + \beta_4^{(n)}(\ell)]n = \alpha_\ell n$$

Achievable Scheme for $(R_1, \ldots, R_L)$



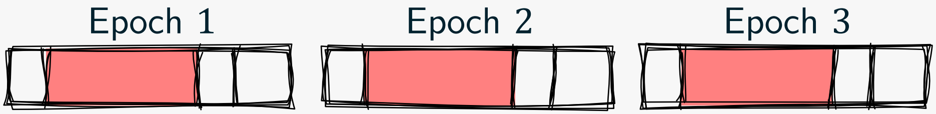
- Training phase
- Message phase
- Retraining phase
- Control phase

Achievable Scheme for $(R_1, \ldots, R_L)$



- Training phase
  - Send training seq of length $\beta_1^{(n)} n$.
  - Channel estimate $\hat{L}_m^{(k,n)}$ using estimation rule $\hat{\theta}_m^{(n)}$
- Message phase
- Retraining phase
- Control phase

Achievable Scheme for $(R_1, \ldots, R_L)$

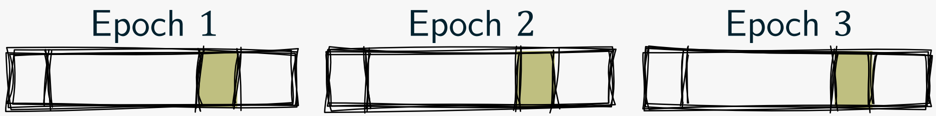


- Training phase
- Message phase

  Encoder and decoder agree on L codebooks

  - Codebook ℓ: No feedback comm over DMC Q_ℓ
    block length $\beta_2^{(n)}(\ell)$; size $M_\ell(n) = \lfloor 2^{n\alpha_\ell \gamma_\ell C_\ell} \rfloor$ (rate $\approx C_\ell$)
  - Use codebook $\hat{L}_m^{(k,n)}$
- Retraining phase
- Control phase

Achievable Scheme for $(R_1, \ldots, R_L)$

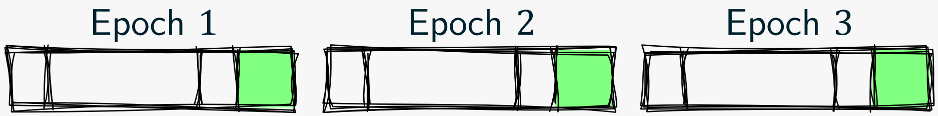


- Training phase
- Message phase
- Retraining phase
  - Send training seq of length $\beta_3^{(n)} n$.
  - Channel estimate $\hat{L}_c^{(k,n)}$ using estimation rule $\hat{\theta}_c^{(n)}$

> $\hat{L}_c^{(k,n)}$ only depends on training sequence of phase 3 of epoch k

- Control phase

Achievable Scheme for $(R_1, \ldots, R_L)$



- Training phase
- Message phase
- Retraining phase
- Control phase

Encoder and decoder agree upon

- L pairs of input symbols $(x_A(\ell), x_R(\ell))$ that are arg max of

$$\max_{x_A, x_R} D(Q_\ell(\cdot|x_A) \| Q_\ell(\cdot|x_R))$$

- L decoding regions $\boldsymbol{\mathcal{A}}_\ell \subset \boldsymbol{\mathcal{Y}}^{\beta_4^{(n)}(\ell)n}$ that optimally distinguish

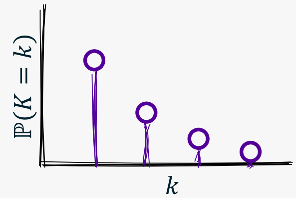
$$\underbrace{(x_A(\ell), \ldots, x_A(\ell))}_{\beta_4^{(n)}(\ell)\text{times}} \quad \text{from} \quad \underbrace{(x_R(\ell), \ldots, x_R(\ell))}_{\beta_4^{(n)}(\ell)\text{times}} \quad \text{over DMC } Q_\ell$$

- Send ACCEPT or REJECT symbols for channel $\hat{L}_c^{(k,n)}$

Performance Analysis

- Number of epochs $K(n)$

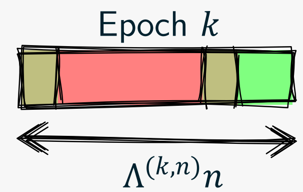
$$\mathbb{P}_\ell(K^{(n)} = k) = p_\ell(n)(1 - p_\ell(n))^n, \quad \lim_{n\to\infty} p_\ell(n) = 1$$



- Relative length of an epoch

$$\Lambda^{(k,n)} = \beta_1^{(n)} + \beta_2^{(n)}(\hat{L}_m^{k,n)}) + \beta_3^{(n)} + \beta_4^{(n)}(\hat{L}_c^{(k,n)})$$

$$\mathbb{E}_\ell[\Lambda^{(k,n)}] \to \alpha_\ell$$



- Expected communication length

$$\mathbb{E}_\ell[\tau^{(n)}] = \mathbb{E}_\ell[K^{(n)}\Lambda^{(K^{(n)},n)}n] \approx \alpha_\ell n$$

Performance Analysis

- Rate

$$\lim_{n\to\infty} \frac{\mathbb{E}_\ell[\log M_{\hat{L}_m^{(k,n)}}(n)]}{\mathbb{E}_\ell[\tau^{(n)}]}$$

$$M_\ell(n) = \lfloor 2^{n\alpha_\ell \gamma_\ell C_\ell} \rfloor$$

Performance Analysis

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$$\lim_{n\to\infty} \frac{\mathbb{E}_\ell[\log M_{\hat{L}_m^{(k,n)}}(n)]}{\mathbb{E}_\ell[\tau^{(n)}]} \approx \frac{n\alpha_\ell\gamma_\ell C_\ell}{\alpha_\ell n} = \gamma_\ell C_\ell$$

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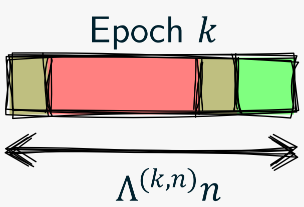
- Error Exponent

$$E_\ell = \lim_{n\to\infty} -\frac{\log P_\ell^{(n)}}{\mathbb{E}_\ell[\tau^{(n)}]}$$

$$\approx \lim_{n\to\infty} -\frac{\text{log(decoding error)} + \text{log(hypothesis testing error)}}{\alpha_\ell n}$$

$$\geq \lim_{n\to\infty} -\frac{\text{log(hypothesis testing error)}}{\alpha_\ell n}$$



Performance Analysis

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$$\lim_{n\to\infty} \frac{\mathbb{E}_\ell[\log M_{\hat{L}_m^{(k,n)}}(n)]}{\mathbb{E}_\ell[\tau^{(n)}]} \approx \frac{n\alpha_\ell\gamma_\ell C_\ell}{\alpha_\ell n} = \gamma_\ell C_\ell = R_\ell$$

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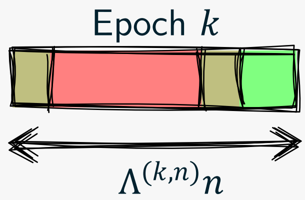
$$\begin{aligned} E_\ell &= \lim_{n\to\infty} -\frac{\log P_\ell^{(n)}}{\mathbb{E}_\ell[\tau^{(n)}]} \\ &\approx \lim_{n\to\infty} -\frac{\text{log(decoding error)} + \text{log(hypothesis testing error)}}{\alpha_\ell n} \\ &\geq \lim_{n\to\infty} -\frac{\text{log(hypothesis testing error)}}{\alpha_\ell n} \\ &\approx -\log\left(e^{-\beta_3^{(n)} n(\square)} + e^{-\beta_4^{(n)}(\ell) n(\square)}\right) \Big/ (\alpha_\ell n) \end{aligned}$$



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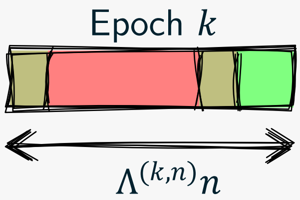

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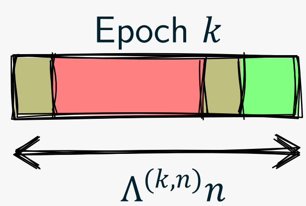
$$\geq \frac{\left({\color{olive}\blacksquare}\right)\cdot\left({\color{green}\blacksquare}\right)}{\left({\color{olive}\blacksquare}\right)+\left({\color{green}\blacksquare}\right)} \frac{\beta_3^{(n)} + \beta_4^{(n)}(\ell)}{\alpha_\ell} = \frac{{\color{olive}T_\ell^c} \cdot {\color{green}B_{Q_\ell}}}{{\color{olive}T_\ell^c} + {\color{green}B_{Q_\ell}}} {\color{red}\left(1 - \frac{R_\ell}{C_\ell}\right)}$$



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$$\lim_{n\to\infty} \frac{\mathbb{E}_\ell[\log M_{\hat{L}_m^{(k,n)}}(n)]}{\mathbb{E}_\ell[\tau^{(n)}]} \approx \frac{n\alpha_\ell\gamma_\ell C_\ell}{\alpha_\ell n} = \gamma_\ell C_\ell = R_\ell$$

$$M_\ell(n) = \lfloor 2^{n\alpha_\ell\gamma_\ell C_\ell} \rfloor$$

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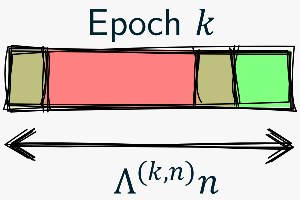
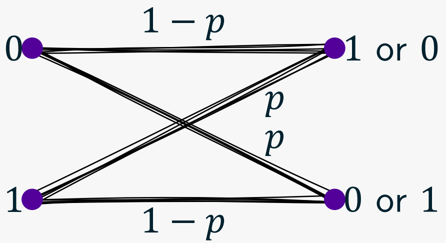


An Example

- $\boldsymbol{Q} = \{\mathsf{BSC}_p, \mathsf{BSC}_{1-p}\}$, p known
- Slope of Burnashev Exponent:

$$B_p = B_{1-p} = D(p \| 1-p)$$



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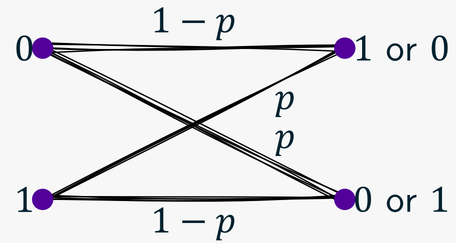
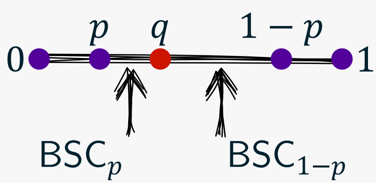


- Slope of Burnashev Exponent:

$$B_p = B_{1-p} = D(p \| 1 - p)$$

- Channel Estimation
  - Transmit all zero sequence
  - Freq of ones $< q$: estimate BSC_p



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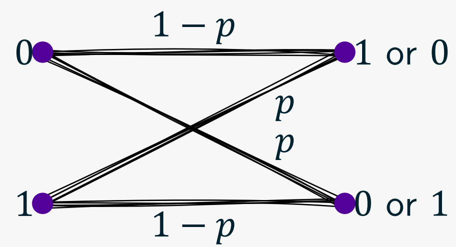

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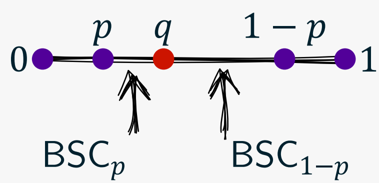


- Exponent of training error

$$T_p = D(p\|q), \quad T_{1-p} = D(1-p\|q)$$

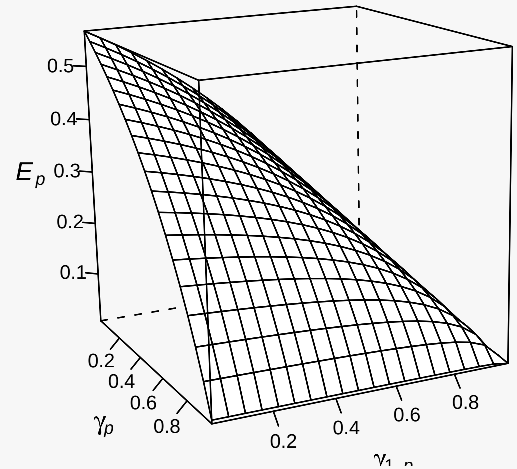
Performance evaluation

Rate $\boldsymbol{R} = (R_p, R_{1-p})$. Let $\gamma = R/C$.

- Error exponents

$$E_p \geq \frac{D(q|p) \cdot D(p|1-p)}{D(q|p) + D(p|1-p)}(1-\gamma_p)$$

$$E_{1-p} \geq \frac{D(q|1-p) \cdot D(p|1-p)}{D(q|1-p) + D(p|1-p)}(1-\gamma_{1-p})$$



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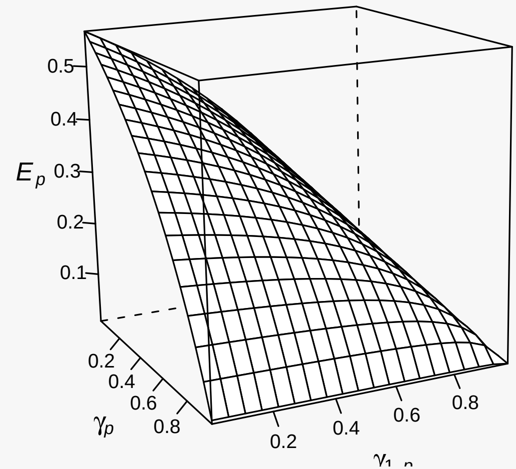


- **Error exponents**

$$E_p \geq \frac{D(q|p) \cdot D(p|1-p)}{D(q|p) + D(p|1-p)}(1-\gamma_p)$$

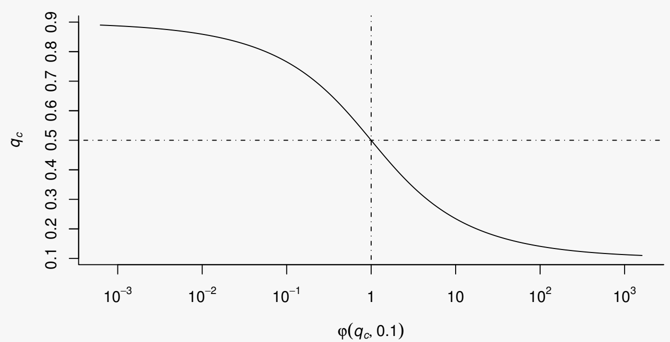
$$E_{1-p} \geq \frac{D(q|1-p) \cdot D(p|1-p)}{D(q|1-p) + D(p|1-p)}(1-\gamma_{1-p})$$

- **Optimal threshold q**

Choose q such that $E_p = E_{1-p}$: solve for q in

$$\varphi(q,p) = \frac{(1-\gamma_p)}{(1-\gamma_{1-p})}$$

where $\varphi(q,p)$ is appropriately defined



Conclusion

- Contributions
  - Defining opportunistic capacity and corresponding error exponent regions for compound channels with feedback.
  - A simple and easy to implement coding scheme whose error exponents are within a multiplicative factor of the best possible error exponents
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- Future directions
  - Channels defined over continuous families and continuous alphabets
  - Upper bound on error exponents

Thank you